

Resilient Consensus of Switched Multi-agent Systems

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Abstract

This letter considers the resilient consensus problem for switched multi-agent systems composed of continuous-time and discrete-time subsystems. We propose a switched filtering strategy for cooperative nodes based upon available local information, withstanding the threat of non-cooperative nodes. We provide conditions that guarantee resilient consensus in the presence of locally bounded Byzantine nodes in directed networks under arbitrary switching. Resilient scaled consensus and resilient scaled formation generation problems for switched multi-agent systems are solved as generalizations. Simulations are also provided to illustrate the effectiveness of the theoretical results.

Keywords:

Resilient consensus, switched multi-agent system, directed network, scaled consensus, formation control.

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1. Introduction

Consensus problem of multi-agent systems has attracted significant attention from diverse contexts due to its broad applications in areas such as distributed computation, sensor networks, cooperative coordination of unmanned aerial vehicles and robotic teams. In general, the main goal of consensus problems is to reach agreement collectively on global quantities of interest, such as the centroid of the network or the average temperature of the environment, using nearest-neighbor rules due to the limited communication capability of each agent in the network [1]. For this purpose, a variety of consensus algorithms have been proposed for both continuous-time and discrete-time systems [2, 3, 4, 5].

Modern large-scale cyber-physical systems are susceptible to failure when one or more nodes are compromised and become non-cooperative. This may be due to malicious attacks (e.g., an attacker taking control of the communication module of certain agents trying to manipulate the whole network) or platform-level failures (e.g., a faulty robot sharing an incorrect location due to a defective GPS sensor). As such, resilience of consensus, and more generally, information diffusion, in the presence of non-cooperative nodes has received increasing attention from control system community in the last few decades [6, 7]. Recent remarkable efforts include a novel definition of network robustness introduced by Zhang et al. and LeBlanc et al. [8, 9], termed r -robustness, which facilitates purely local interaction rules for resilient consensus against malicious nodes. The proposed update approach is able to deal with the situation where the identities

and actual number of non-cooperative nodes are unknown to the cooperative nodes in the network. The results have been later generalized to the case of hybrid dynamics [10] as well as second-order multi-agent systems with discretized dynamics [11] under locally bounded faults. Tolerance to Byzantine behaviors and communication delay is further investigated in [12] for a continuous-time multi-agent system. Furthermore, resilient flocking of mobile robot teams in the presence of non-cooperative robots has been tackled in [13]. It is worth mentioning that resilience to non-cooperative nodes is conceptually different from resilience to disturbance or noise [14, 15], and the methods used are distinct.

The results of all previous works on resilient consensus, to our knowledge, are concerned with multi-agent systems composed of either only continuous-time systems or only discrete-time systems. In reality, a large-scale system can be split into multiple subsystems, and a switching rule manages the switching between them leading to a switched multi-agent system [16]. Such switching behavior exists in the dynamical behavior of agents rather than merely in the communication topology, the latter of which has been extensively studied in multi-agent systems with switching topology; see e.g. [1, 5]. A continuous-time plant, for instance, is controlled either by a digitally implemented regulator or a physically implemented one with a synchronous switching law between them. For such a continuous-time multi-agent system, if all the agents are activated by using computers at discrete time steps, then the whole system can be regarded as a switched system containing both continuous-time and discrete-time subsystems. In fact, switched multi-agent systems are often generated in a computer-aided system, where computers are used to activate all the continuous-time agents in a discretized manner. Consensus problems for switched multi-agent systems are firstly considered in [17] over varied topologies using Lyapunov function. Ref. [18] studies the finite-time convergence for switched multi-agent systems based on finite-time stability theory. Scaled consensus and containment control of switched multi-agent systems are also addressed in [19] and [20], respectively.

Here, we focus on resilient consensus of switched multi-agent systems consisting of both continuous-time and discrete-time subsystems. We aim to design appropriate consensus protocols to withstand the compromise of a subset of the nodes and reach the group objective in a purely distributed manner. The main contribution of this letter is threefold. First, a consensus protocol is proposed for the switched multi-agent systems. Sufficient conditions are presented to guarantee resilient consensus in the presence of locally bounded Byzantine nodes under arbitrary switching. The proof of our main result (Theorem 1 below) is carefully divided into two cases, in each of which either the continuous-time subsystem or the discrete-time subsystem dominates in a sense. Moreover, unlike the work [17, 18, 19, 20], no complicated algebraic conditions related to eigenvalues or eigenvectors are imposed here. Second, as an extension, resilient scaled consensus problems are introduced and solved for switched multi-agent systems. In scaled consensus problems, agents' states reach asymptotically assigned ratios in terms of possibly different scales instead of a common value [21, 22]. They can be specialized to achieve standard consensus, cluster consensus, bipartite consensus, etc. Third, resilient (scaled) formation generation and formation tracking problems are also solved as further generations.

The rest of the letter is organized as follows. Section 2 provides some preliminaries and formulates the resilient consensus problems of switched multi-agent systems. The main results are given in Section 3. We provide a simulation example in Section 4 and then conclude the article in Section 5.

2. Preliminaries

2.1. Graph theory

Denote by \mathbb{R} and \mathbb{N} the sets of reals and non-negative integers, respectively. With some ambiguity, let $|\cdot|$ be the cardinality of a set (note that we do not consider multi-sets in this paper) and also the absolute value of a real. The actual meaning will be clear from the context. A directed graph (digraph) of order n is denoted by $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{v_1, \dots, v_n\}$ is the node set with $|\mathcal{V}| = n$ representing the agents in the network, and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the directed edge set. We consider a partition of the node set, $\mathcal{V} = \mathcal{C} \cup \mathcal{B}$, where \mathcal{C} is the set of cooperative nodes and \mathcal{B} is the set of non-cooperative nodes which is unknown a priori to the cooperative nodes. An edge (v_i, v_j) describes the information flow from node v_i to node v_j . The neighborhood of node v_i is defined by $\mathcal{N}_i = \{v_j : (v_j, v_i) \in \mathcal{E}\}$. A directed path from node v_{i_1} to v_{i_l} is a sequence of edges $(v_{i_1}, v_{i_2}), (v_{i_2}, v_{i_3}), \dots, (v_{i_{l-1}}, v_{i_l})$ in \mathcal{G} with distinct nodes $v_{i_k}, k = 1, 2, \dots, l$. \mathcal{G} is said to contain a directed spanning tree with root node v_r if for every node $v \in \mathcal{V}$ except v_r , there exists a directed path from v_r to v .

The following notions of reachable set and network robustness are introduced in [8, 9, 23], which have close relationship with conventional graph-theoretic connectivity and play a key role in resilient coordination.

Definition 1. (reachable set) Let $r \in \mathbb{N}$. A set $S \subseteq \mathcal{V}$ is an r -reachable set if there exists a node $v_i \in S$ such that $|\mathcal{N}_i \setminus S| \geq r$.

Definition 2. (network robustness) Let $r \in \mathbb{N}$. A digraph \mathcal{G} is r -robust if for any pair of nonempty, disjoint subsets of \mathcal{V} , at least one of them is r -reachable.

Lemma 1. [9] *Given an r -robust digraph \mathcal{G} , let \mathcal{G}' be the graph produced by removing up to s incoming edges of each node in \mathcal{G} , where $0 \leq s < r$. Then \mathcal{G}' is $r - s$ -robust. Moreover, a digraph \mathcal{G} is 1-robust if and only if it contains a directed spanning tree.*

2.2. Model description

Consider a group of n agents, $\{v_1, \dots, v_n\}$, composing a directed communication network $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with $\mathcal{V} = \mathcal{C} \cup \mathcal{B}$. The agents are controlled by a switching law between continuous-time dynamics and discrete-time dynamics. The information state of the agent v_i at time t is represented by $x_i(t) \in \mathbb{R}$. We are devoted to addressing the following resilient consensus problem.

Definition 3. (resilient consensus) We say that the cooperative agents in \mathcal{C} achieve resilient consensus in the presence of non-cooperative nodes in \mathcal{B} if $\lim_{t \rightarrow \infty} x_i(t) - x_j(t) = 0$ for all $v_i, v_j \in \mathcal{C}$ and all initial conditions $\{x_i(0)\}_{v_i \in \mathcal{V}}$.

Obviously, when $\mathcal{B} = \emptyset$, the definition reduces to the standard consensus. The dynamical model of each cooperative node $v_i \in \mathcal{C}$ is described by a continuous-time subsystem

$$\dot{x}_i(t) = f_i^C \left(\{x_j^i(t) : v_j \in \mathcal{N}_i \cup \{v_i\}\} \right) \quad (1)$$

and a discrete-time subsystem

$$x_i(t+1) = f_i^D \left(\{x_j^i(t) : v_j \in \mathcal{N}_i \cup \{v_i\}\} \right) \quad (2)$$

where $x_j^i(t) \in \mathbb{R}$ is the value sent from node v_j to node v_i at time t , and $x_j^i(t) = x_j(t)$ for $v_j \in \mathcal{C}$. By definition we have $x_i^i(t) = x_i(t)$. Here, f_i^C and f_i^D describe the update functions for cooperative node v_i , which are to be designed so that the cooperative nodes could achieve the system's

objective withstanding the compromise of non-cooperative nodes, whose identities and number remain unknown. Non-cooperative nodes, on the other hand, may apply different, arbitrary update rules that are not known to the cooperative nodes. Specifically, we consider the Byzantine nodes in this letter.

Definition 4. (Byzantine node) A node $v_i \in \mathcal{B}$ is said to be Byzantine if it applies some different update rule \tilde{f}_i^C for the continuous-time subsystem or \tilde{f}_i^D for the discrete-time subsystem, or it does not send the same value to all of its neighbors at some time $t > 0$.

Byzantine nodes are often regarded as the worst-case attackers [9, 10, 12], who usually have a complete knowledge of the whole system and send arbitrary wrong information to its neighbors either through broadcast communication or in a point-to-point manner. According to the number and location of the non-cooperative nodes, globally bounded and locally bounded models have been extensively studied in the literature; see e.g. [8, 9, 10, 11, 12]. Let $R \in \mathbb{N}$. We here consider the R -locally bounded model, where $|\mathcal{N}_i \cap \mathcal{B}| \leq R$ for each $v_i \in \mathcal{C}$. In the R -locally bounded model, each cooperative node has at most R compromised neighbors, which threaten the group objective by preventing other agents from achieving valid states or driving their values into an unsafe region. Therefore, it is desirable to design resilient consensus strategies.

Here, we adopt the nearest-neighbor rules and construct distributed local filtering algorithms for each cooperative node $v_i \in \mathcal{C}$ based on the Weighted-Mean-Subsequence-Reduced (W-MSR) algorithm [9, 12] as follows.

For continuous-time subsystem, the filtering algorithm consists of three steps, executed at time $t \in \mathbb{R}$. Fix $R \in \mathbb{N}$. First, each cooperative node $v_i \in \mathcal{C}$ obtains the values $\{x_j^i(t)\}$ of its neighbors, and creates a sorted list for $\{x_j^i(t)\}_{v_j \in \mathcal{N}_i}$ from largest to smallest. Second, the largest R values that are strictly larger than $x_i(t)$ in this list are removed (if there are fewer than R larger values than $x_i(t)$, all of those values are removed). The similar removal process is applied to the smaller values. The set of nodes that are removed by node v_i at time t is denoted by $\mathcal{R}_i(t)$. Third, each $v_i \in \mathcal{C}$ updates its value using the following $f_i^C(\cdot)$ in (1):

$$\dot{x}_i(t) = \sum_{v_j \in (\mathcal{N}_i \cup \{v_i\}) \setminus \mathcal{R}_i(t)} \varphi_{ij}(x_j^i(t), x_i(t)), \quad (3)$$

where the function $\varphi_{ij} : \mathbb{R}^2 \rightarrow \mathbb{R}$ satisfies (iC) φ_{ij} is locally Lipschitz continuous, (iiC) $\varphi_{ij}(x, y) = 0$ if and only if $x = y$, and (iiiC) $(x - y)\varphi_{ij}(x, y) > 0$ for any $x \neq y$.

For discrete-time subsystem, the filtering algorithm consists of three steps, executed at each time step $\hat{t} + t$, where $t \in \mathbb{N}$ and $\hat{t} \in \mathbb{R}$ signifies the time at which the switch from a continuous-time subsystem to a discrete-time subsystem takes place. Fix $R \in \mathbb{N}$. First, each cooperative node $v_i \in \mathcal{C}$ obtains the values $\{x_j^i(t)\}$ of its neighbors, and creates a sorted list for $\{x_j^i(t)\}_{v_j \in \mathcal{N}_i}$ from largest to smallest. Second, the largest R values that are strictly larger than $x_i(t)$ in this list are removed (if there are fewer than R larger values than $x_i(t)$, all of those values are removed). The similar removal process is applied to the smaller values. The set of nodes that are removed by node v_i at time t is denoted by $\mathcal{R}_i(t)$. Third, each $v_i \in \mathcal{C}$ updates its value using the following $f_i^D(\cdot)$ in (2):

$$x_i(t+1) = \sum_{v_j \in (\mathcal{N}_i \cup \{v_i\}) \setminus \mathcal{R}_i(t)} w_{ij}(t)x_j^i(t), \quad (4)$$

where $w_{ij}(t)$ are the weights satisfying (iD) $w_{ij}(t) = 0$ if $v_j \notin \mathcal{N}_i \cup \{v_i\}$, (iiD) there exists a constant $\alpha \in (0, 1)$ independent of t , such that $w_{ij}(t) \geq \alpha > 0$ for any $v_j \in (\mathcal{N}_i \cup \{v_i\}) \setminus \mathcal{R}_i(t)$, and (iiiD) $\sum_{v_j \in (\mathcal{N}_i \cup \{v_i\}) \setminus \mathcal{R}_i(t)} w_{ij}(t) = 1$.

Remark 1. Note that the functions φ_{ij} in (3) and the weights $w_{ij}(t)$ in (4) can be arbitrarily chosen as long as the corresponding conditions hold. A typical choice for continuous-time subsystems is $\varphi_{ij}(x, y) = a_{ij}(x - y)$, where $a_{ij} > 0$ indicates the coupling strength of edge (v_i, v_j) ; see e.g., [1, 3]. For discrete-time subsystems, we can simply take $w_{ij}(t) = (|\mathcal{N}_i| + 1 - |\mathcal{R}_i(t)|)^{-1}$ so that each neighbor has the same weight.

The above algorithm has low complexity and is purely distributed using only local information. No prior knowledge of the identities of non-cooperative nodes or the network topology is assumed for cooperative nodes. At time instant t , the choice of subsystem is determined by the switching rule under consideration. The switch between continuous-time control and sampled-data control guarantees the switched multi-agent system (1)-(4) to be composed of continuous-time and discrete-time subsystems (or only the continuous-time subsystems, or only the discrete-time subsystems). In what follows, we will refer to the above algorithm as the switched filtering strategy with parameter R .

Remark 2. The above filtering strategy is reminiscent of opinion dynamics, where opinion spreading models with bounded confidence are widely used to explain the consensus of individual's opinions by ignoring the opinions that are quite different from one's own while updating his or her opinion [24, 25]. A key difference here is that in these opinion dynamics models, the neighborhood of a node is formed based upon the opinion differences specified by a given threshold and the number of deleted nodes is indefinite. Nevertheless, in our resilient consensus framework, the neighborhood of a node is not shaped by a fixed threshold and the number of deleted nodes has a given upper bound.

3. Main results

In this section, we study the resilient consensus of switched multi-agent system (1)-(4) in the R -locally bounded model with Byzantine nodes. To begin with, define $M(t) := \max_{v_i \in C} x_i(t)$ and $m(t) := \min_{v_i \in C} x_i(t)$ as the maximum and minimum values within cooperative nodes at time t .

Lemma 2. Consider the switched multi-agent system (1)-(4) over the digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where each cooperative node performs the switched filtering strategy with parameter R . In the R -locally bounded model with Byzantine nodes, for each node $v_i \in C$, we have $x_i(t+1) \in [m(t), M(t)]$ if the discrete-time subsystem is activated on $[t, t+1]$, and $x_i(t) \in [m(0), M(0)]$ if the continuous-time subsystem is activated at time t , regardless of the switching rule.

Proof. Fix $v_i \in C$. When the discrete-time subsystem (4) is activated on the time interval $[t, t+1]$, the value $x_i(t+1)$ is a convex combination of values $\{x_j^i(t)\}_{j \in (\mathcal{N}_i \cup \{v_i\}) \setminus \mathcal{R}_i(t)}$, which lie in the interval $[m(t), M(t)]$ according to the switched filtering strategy with parameter R and the definition of R -locally bounded model with Byzantine nodes. Therefore, $x_i(t+1) \in [m(t), M(t)]$.

When the continuous-time subsystem (3) is activated at time t , we will only show $x_i(t) \leq M(0)$ (the analogous inequality $x_i(t) \geq m(0)$ can be shown similarly). If this is not true, there exists some time $t^* < t$, at which the continuous-time subsystem is activated, such that (a) $x_j(t^*) \leq M(0)$ for any $t^* \leq t$ and any $v_j \in C$ and (b) $x_i(t^*) = M(0)$ and $\dot{x}_i(t^*) > 0$. This is because $M(t)$ is monotonically decreasing whenever the discrete-time subsystem is activated. Now we have

$$\dot{x}_i(t^*) = \sum_{v_j \in (\mathcal{N}_i \cup \{v_i\}) \setminus \mathcal{R}_i(t^*)} \varphi_{ij}(x_j^i(t^*), x_i(t^*)). \quad (5)$$

Note that $x_i(t^*) = M(0) \geq M(t^*) \geq x_j^i(t^*)$, where the last inequality holds due to the switched filtering strategy with parameter R and the fact that there are at most R Byzantine nodes in \mathcal{N}_i . In

view of (iiC) and (iiiC), each term on the right-hand side of (5) is non-positive. Thus, $\dot{x}_i(t^*) \leq 0$, which is a contradiction. \square

Remark 3. It follows from Lemma 2 that the interval $[m(0), M(0)]$ containing the initial values of the cooperative nodes is an invariant set, implying that the final consensus value would be within this interval if resilient consensus is achieved for the switched multi-agent system. This property is essential for some safety critical processes, where the interval $[m(0), M(0)]$ is known to be safe.

Note that although the communication topology \mathcal{G} is fixed, the set $\mathcal{R}_i(t)$ in (3) and (4) is time-dependent, which makes the network topology time-dependent essentially. We make the following assumption for the continuous-time subsystem.

Assumption 1. Denote by $\{\tau_l\}_{l \in \mathbb{N}}$ the sequence of time points at which $\mathcal{R}_i(t)$ changes for some $v_i \in C$ and the continuous-time subsystem is activated. Assume that $|\tau_{l+1} - \tau_l| \geq \tau > 0$ for some constant $\tau > 0$ or $\tau = \infty$.

When τ is infinity, it means that from the time $t = \tau_l$ on the network topology remains unchanged for the continuous-time subsystem. Assumption 1 reflects the boundedness of dwell time $\tau_{l+1} - \tau_l$, which is often instrumental in dealing with continuous-time multi-agent systems with switching topologies; see e.g. [1]. Our first result regarding resilient consensus of switched system is the following.

Theorem 1. Consider the switched multi-agent system (1)-(4) over the digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where each cooperative node performs the switched filtering strategy with parameter R . Suppose \mathcal{G} is $2R+1$ -robust and Assumption 1 holds. Then, in the R -locally bounded model with Byzantine nodes, resilient consensus is achieved under arbitrary switching.

Proof. Without loss of generality, we assume there is a sequence of time instants $0 \leq t_1 \leq \bar{t}_1 \leq t_2 \leq \bar{t}_2 \leq \dots \leq t_k \leq \bar{t}_k \leq \dots$ such that the continuous-time subsystem (3) is activated when $t \in (t_k, \bar{t}_k]$ and the discrete-time subsystem (4) is activated when $t \in (\bar{t}_{k-1}, t_k]$. We will prove the theorem in the following two complementary cases: (a) $\lim_{k \rightarrow \infty} \bar{t}_k - t_k = 0$ and (b) there exists $k_1 \in \mathbb{N}$ and $\Delta > 0$ such that $\bar{t}_k - t_k \geq \Delta$ for all $k \geq k_1$.

Case (a). Since $\lim_{k \rightarrow \infty} \bar{t}_k - t_k = 0$ and $x_i(t)$ for $v_i \in C$ is continuous when the continuous-time subsystem is activated, we may assume that $\rho_M := \lim_{t \rightarrow \infty} M(t) \geq \rho_m := \lim_{t \rightarrow \infty} m(t)$ in view of Lemma 2. If $\rho_M = \rho_m$, resilient consensus is reached. In what follows, we assume that $\rho_M > \rho_m$ and will prove that it can not be true by contradiction.

Choose $\varepsilon_0 > 0$ satisfying $\rho_M - \varepsilon_0 > \rho_m + \varepsilon_0$. For $t > 0$ and $\varepsilon_s > 0$, we define two sets $A_M(t, \varepsilon_s) := \{v_i \in C : x_i(t) > \rho_M - \varepsilon_s\}$ and $A_m(t, \varepsilon_s) := \{v_i \in C : x_i(t) < \rho_m + \varepsilon_s\}$. By the definition of ε_0 , $A_M(t, \varepsilon_0)$ and $A_m(t, \varepsilon_0)$ are disjoint. Fix $\varepsilon < \frac{2\alpha^{|\mathcal{C}|}\varepsilon_0}{3(1-\alpha^{|\mathcal{C}|})}$ which satisfies $\varepsilon_0 > \varepsilon > 0$. Recall that $\alpha \in (0, 1)$ is given in condition (iiD). Since $\lim_{k \rightarrow \infty} \bar{t}_k - t_k = 0$ and $x_i(t)$ for $v_i \in C$ is continuous when the continuous-time subsystem is activated, there exists some $k_0 \in \mathbb{N}$ such that for all $t \geq \bar{t}_{k_0}$ and all $v_i \in C$, we have $x_i(t+1) \in [m(t) - \varepsilon/2, M(t) + \varepsilon/2]$ regardless of the subsystems activated on $[t, t+1]$. Let $t_\varepsilon \geq \bar{t}_{k_0}$ be the time step such that $M(t) < \rho_M + \varepsilon$ and $m(t) > \rho_m - \varepsilon$ for all $t \geq t_\varepsilon$.

Define $\mathcal{G}_C = (C, \mathcal{E}_C)$ as the subgraph of $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ induced by C , where \mathcal{E}_C consists of all directed edges among the cooperative nodes at time t . Since \mathcal{G} is $2R+1$ -robust and there are at most R Byzantine neighbors for each cooperative node, \mathcal{G}_C must be $R+1$ -robust. Consider the nonempty, disjoint sets $A_M(t_\varepsilon, \varepsilon_0)$ and $A_m(t_\varepsilon, \varepsilon_0)$. There exists a node in $A_M(t_\varepsilon, \varepsilon_0)$ or in $A_m(t_\varepsilon, \varepsilon_0)$ that has at least $R+1$ cooperative neighbors outside of its set. Without loss of generality, we assume that $v_i \in A_M(t_\varepsilon, \varepsilon_0)$ has at least $R+1$ cooperative neighbors outside of $A_M(t_\varepsilon, \varepsilon_0)$. When the discrete-time subsystem is activated on $[t_\varepsilon, t_\varepsilon + 1]$, noting that these neighbors' values are at

most equal to $\rho_M - \varepsilon_0$ and at least one of these values will be used by v_i , we obtain

$$\begin{aligned} x_i(t_\varepsilon + 1) &\leq (1 - \alpha) \left(M(t_\varepsilon) + \frac{\varepsilon}{2} \right) + \alpha(\rho_M - \varepsilon_0) \\ &\leq \rho_M - \alpha\varepsilon_0 + \frac{3(1 - \alpha)}{2}\varepsilon, \end{aligned} \quad (6)$$

where we have used the inequality $M(t_\varepsilon) \leq \rho_M + \varepsilon$, and the fact that each cooperative node's value is a convex combination of its own value and the values of its neighbors with coefficients bounded below by α and that the largest value v_i will use at time t_ε is upper bounded by $M(t_\varepsilon) + \varepsilon/2$ according to the switched filtering strategy with parameter R . If the continuous-time subsystem is activated on some time interval $I \subset [t_\varepsilon, t_\varepsilon + 1]$, it is easy to see that (6) remains valid by choosing t_ε large enough since $|I|$ is vanishing in Case (a). Therefore, the inequality (6) holds regardless of the subsystems activated on $[t_\varepsilon, t_\varepsilon + 1]$. Moreover, the inequality (6) also applies to the updated value of any cooperative node outside $A_M(t_\varepsilon, \varepsilon_0)$ since such a node will use its own value (which is also upper bounded by $\rho_M - \varepsilon_0$) in the update procedure. Likewise, if $v_i \in A_m(t_\varepsilon, \varepsilon_0)$ which has at least $R + 1$ cooperative neighbors outside of $A_m(t_\varepsilon, \varepsilon_0)$, we obtain a similar bound

$$x_i(t_\varepsilon + 1) \geq \rho_m + \alpha\varepsilon_0 - \frac{3(1 - \alpha)}{2}\varepsilon, \quad (7)$$

which also applies to the cooperative nodes outside $A_m(t_\varepsilon, \varepsilon_0)$.

Set $\varepsilon_1 = \alpha\varepsilon_0 - 3(1 - \alpha)\varepsilon/2$, which satisfies $0 < \varepsilon < \varepsilon_1 < \varepsilon_0$. Note that the sets $A_M(t_\varepsilon + 1, \varepsilon_1)$ and $A_m(t_\varepsilon + 1, \varepsilon_1)$ are disjoint. The discussion in the above paragraph implies that $|A_M(t_\varepsilon + 1, \varepsilon_1)| < |A_M(t_\varepsilon, \varepsilon_0)|$ or $|A_m(t_\varepsilon + 1, \varepsilon_1)| < |A_m(t_\varepsilon, \varepsilon_0)|$ holds. We can recursively define $\varepsilon_s = \alpha\varepsilon_{s-1} - 3(1 - \alpha)\varepsilon/2$ for each $k \geq 1$ and note that $\varepsilon_s < \varepsilon_{s-1}$. The above discussion can be applied to each time step $t_\varepsilon + s$ as long as $A_M(t_\varepsilon + s, \varepsilon_s)$ and $A_m(t_\varepsilon + s, \varepsilon_s)$ are still non-empty. Since there are $|\mathcal{C}|$ cooperative nodes in \mathcal{G}_C , there exists some $T \leq |\mathcal{C}|$ such that either $A_M(t_\varepsilon + T, \varepsilon_T)$ or $A_m(t_\varepsilon + T, \varepsilon_T)$ is empty. On the other hand, $\varepsilon_T = \alpha\varepsilon_{T-1} - 3(1 - \alpha)\varepsilon/2 = \alpha^T \varepsilon_0 - 3(1 - \alpha^T)\varepsilon/2 \geq \alpha^{|\mathcal{C}|} \varepsilon_0 - 3(1 - \alpha^{|\mathcal{C}|})\varepsilon/2 > 0$ by our choice of ε . This means that all cooperative nodes at time $t_\varepsilon + T$ have values at most $\rho_M - \varepsilon_T < \rho_M$ or have values at least $\rho_m + \varepsilon_T > \rho_m$. This contradicts the definition of ρ_M or ρ_m , completing the proof of Case (a).

Case (b). Define the Dini derivative of a function $f(t)$ as $D^+ f(t) = \limsup_{h \rightarrow 0^+} (f(t+h) - f(t))/h$. Fix any $t \geq t_{k_1}$. When $t \in (t_k, \bar{t}_k]$ for some $k \in \mathbb{N}$, the Dini derivatives of $M(t)$ and $m(t)$ along the trajectory of (3) are given by

$$\begin{aligned} D^+ M(t) &= \dot{x}_{i_0}(t) \\ &= \sum_{v_j \in (\mathcal{N}_{i_0} \cup \{v_{i_0}\}) \setminus \mathcal{R}_{i_0}(t)} \varphi_{i_0 j}(x_j^{i_0}(t), x_{i_0}(t)) \end{aligned} \quad (8)$$

and

$$\begin{aligned} D^+ m(t) &= \dot{x}_{i_1}(t) \\ &= \sum_{v_j \in (\mathcal{N}_{i_1} \cup \{v_{i_1}\}) \setminus \mathcal{R}_{i_1}(t)} \varphi_{i_1 j}(x_j^{i_1}(t), x_{i_1}(t)), \end{aligned} \quad (9)$$

where $\dot{x}_{i_0}(t) = \max_{i \in I_0(t)} \dot{x}_i(t)$, $I_0(t) = \{i : x_i(t) = M(t), v_i \in \mathcal{C}\}$ and $\dot{x}_{i_1}(t) = \max_{i \in I_1(t)} \dot{x}_i(t)$, $I_1(t) = \{i : x_i(t) = m(t), v_i \in \mathcal{C}\}$ following the property of Dini derivative; see e.g. [26]. Define

$V(t) = M(t) - m(t)$. Since $x_{i_0}(t) \geq x_j^{i_0}(t)$ in (8) and $x_{i_1}(t) \leq x_j^{i_1}(t)$ in (9), we derive that the right-hand side of (8) is non-positive and that the right-hand side of (9) is non-negative by employing the conditions (iiC) and (iiiC). Hence, $D^+V(t) = D^+M(t) - D^+m(t) \leq 0$. When $t \in (\bar{t}_{k-1}, t_k]$ for some $k \in \mathbb{N}$, it is easy to see that $D^+V(t) = D^+M(t) - D^+m(t) \leq 0$ still holds thanks to the monotonicity shown in Lemma 2.

Next, we prove $\lim_{t \rightarrow \infty} D^+V(t) = 0$ by contradiction. Suppose this is not true. Since the continuous-time subsystem is activated repeatedly in Case (b), there exist constants $\varepsilon_0 > 0$, $\delta_0 > 0$, and a sequence of time points $\{s_p\}_{p \in \mathbb{N}}$, at which the continuous-time subsystem is activated such that (i) s_p tends to infinity as p goes to infinity and (ii) $D^+V(s_p) \leq -\varepsilon_0$ and $|s_{p+1} - s_p| > \delta_0$ hold for all $p \in \mathbb{N}$.

Consider any time interval $I \subset (t_k, \bar{t}_k]$ for some $k \geq k_1$ such that $I \cap \{\tau_l\}_{l \in \mathbb{N}} = \emptyset$. Since $D^+V(t)$ is continuous in I and $\dot{x}_i(t)$ is bounded for all $v_i \in \mathcal{C}$ by condition (iC), we know that $D^+V(t)$ is uniformly continuous in I . Hence, there exists a constant $\delta_1 > 0$ such that for any $t^1, t^2 \in I$ and $|t^1 - t^2| < \delta_1$, we have $|D^+V(t^1) - D^+V(t^2)| < \varepsilon_0/2$. By Assumption 1 and the assumption of Case (b), we may choose $\delta_2 \in (0, \delta_1)$ so that for each $p \in \mathbb{N}$, the interval $[s_p - \delta_2, s_p + \delta_2]$ is contained in some interval I described above. Therefore, for any $t \in [s_p - \delta_2, s_p + \delta_2]$, we obtain

$$\begin{aligned} D^+V(t) &= -|D^+V(s_p) - (D^+V(s_p) - D^+V(t))| \\ &\leq -(|D^+V(s_p)| - |D^+V(s_p) - D^+V(t)|) \\ &\leq -\varepsilon_0 + \frac{\varepsilon_0}{2} = -\frac{\varepsilon_0}{2}. \end{aligned} \quad (10)$$

Recall that $|s_{p+1} - s_p| > \delta_0$ for all $p \in \mathbb{N}$. We can choose $\delta \in (0, \delta_2)$ such that the intervals $\{[s_p - \delta, s_p + \delta]\}_{p \in \mathbb{N}}$ are pairwise disjoint. By (10) and recalling that $D^+V(t) \leq 0$ for $t \geq t_{k_1}$, we derive

$$\begin{aligned} \int_{t_{k_1}}^{\infty} D^+V(t) dt &\leq \lim_{N \rightarrow \infty} \sum_{p=1}^N \int_{s_p - \delta}^{s_p + \delta} D^+V(t) dt \\ &\leq - \lim_{N \rightarrow \infty} \sum_{p=1}^N \int_{s_p - \delta}^{s_p + \delta} \frac{\varepsilon_0}{2} dt \\ &= - \lim_{N \rightarrow \infty} N \varepsilon_0 \delta = -\infty. \end{aligned} \quad (11)$$

Clearly, (11) conflicts with the fact $V(t) = M(t) - m(t) \geq 0$ for all time t , completing the proof of $\lim_{t \rightarrow \infty} D^+V(t) = 0$.

Recall $D^+M(t) \leq 0$ and $D^+m(t) \geq 0$ for $t \geq t_{k_1}$. By employing (8) and (9), the above limit $\lim_{t \rightarrow \infty} D^+V(t) = 0$ implies that $\lim_{t \rightarrow \infty} M(t) = \lim_{t \rightarrow \infty} x_{i_0}(t) = \eta_M$ and $\lim_{t \rightarrow \infty} m(t) = \lim_{t \rightarrow \infty} x_{i_1}(t) = \eta_m$ for some constants η_M and η_m satisfying $\eta_M \geq \eta_m$. Suppose that $\eta_M > \eta_m$. Since \mathcal{G} is $2R + 1$ -robust, by Lemma 1 the communication topology always contains a directed spanning tree when the switched filtering strategy with parameter R is applied. There exists a time point T and a constant $\varepsilon > 0$ such that $x_{i_0}(t) > \eta_M - \varepsilon > \eta_m + \varepsilon > x_{i_1}(t)$ for $t \geq T$. Since $\lim_{t \rightarrow \infty} \dot{x}_{i_0}(t) = 0$, we have $\lim_{t \rightarrow \infty} x_j^{i_0}(t) - x_{i_0}(t) = 0$ for all $v_j \in (\mathcal{N}_{i_0} \cup \{v_{i_0}\}) \setminus \mathcal{R}_{i_0}(t)$ by employing (3), conditions (iiC), (iiiC), the definition of v_{i_0} , and the definition of R -locally bounded model. Likewise, the limit $\lim_{t \rightarrow \infty} \dot{x}_{i_1}(t) = 0$ implies that $\lim_{t \rightarrow \infty} x_j^{i_1}(t) - x_{i_1}(t) = 0$ for all $v_j \in (\mathcal{N}_{i_1} \cup \{v_{i_1}\}) \setminus \mathcal{R}_{i_1}(t)$. Noting that \mathcal{G} is a finite network, there must exist some time point $T' \geq T$, at which the continuous-time subsystem is activated, such that (i) there exist two directed paths, one from the root node v_r to v_{i_0} and the other from v_r to v_{i_1} , in the communication topology

at time T' and (ii) $x_r(T') > \eta_M - \varepsilon$ and $x_r(T') < \eta_m + \varepsilon$. Clearly, (ii) is a contradiction. Thus, we have $\eta_M = \eta_m$, which completes the proof of Case (b). \square

Given scalar scale $\alpha_i \neq 0$ for each node $v_i \in \mathcal{V}$, resilient scaled consensus can be defined as follows.

Definition 5. (resilient scaled consensus) We say that the cooperative agents achieve resilient scaled consensus with respect to $(\alpha_1, \dots, \alpha_n)$ if $\lim_{t \rightarrow \infty} \alpha_i x_i(t) - \alpha_j x_j(t) = 0$ for all $v_i, v_j \in \mathcal{C}$ and all initial conditions $\{x_i(0)\}_{v_i \in \mathcal{V}}$.

It is obvious that we reproduce the resilient consensus (Definition 3) by setting $\alpha_1 = \alpha_2 = \dots = \alpha_n = 1$. Scaled consensus gives desired flexibility in many real-world problems [19, 21, 22]. To realize resilient scaled consensus for switched multi-agent systems, we need to modify our previous filtering algorithm. We propose the following switched scaled filtering strategy with parameter R :

For continuous-time subsystem, the algorithm consists of three steps, executed at time $t \in \mathbb{R}$. Fix $R \in \mathbb{N}$. First, each cooperative node $v_i \in \mathcal{C}$ obtains the values $\{x_j^i(t)\}$ of its neighbors, and creates a sorted list for $\{\alpha_j x_j^i(t)\}_{v_j \in \mathcal{N}_i}$ from largest to smallest. Second, the largest R values that are strictly larger than $\alpha_i x_i(t)$ in this list are removed (if there are fewer than R larger values than $\alpha_i x_i(t)$, all of those values are removed). The similar removal process is applied to the smaller values. The set of nodes that are removed by node v_i at time t is denoted by $\mathcal{R}_i(t)$. Third, each $v_i \in \mathcal{C}$ updates its value using the following $f_i^C(\cdot)$ in (1):

$$\dot{x}_i(t) = \text{sgn}(\alpha_i) \sum_{v_j \in (\mathcal{N}_i \cup \{v_i\}) \setminus \mathcal{R}_i(t)} \varphi_{ij}(\alpha_j x_j^i(t), \alpha_i x_i(t)), \quad (12)$$

where $\text{sgn}(\cdot)$ is the signum function, the function $\varphi_{ij} : \mathbb{R}^2 \rightarrow \mathbb{R}$ satisfies (iC) φ_{ij} is locally Lipschitz continuous, (iiC) $\varphi_{ij}(x, y) = 0$ if and only if $x = y$, and (iiiC) $(x - y)\varphi_{ij}(x, y) > 0$ for any $x \neq y$.

For discrete-time subsystem, the algorithm consists of three steps, executed at each time step $t \in \mathbb{N}$. Fix $R \in \mathbb{N}$. First, each cooperative node $v_i \in \mathcal{C}$ obtains the values $\{x_j^i(t)\}$ of its neighbors, and creates a sorted list for $\{\alpha_j x_j^i(t)\}_{v_j \in \mathcal{N}_i}$ from largest to smallest. Second, the largest R values that are strictly larger than $\alpha_i x_i(t)$ in this list are removed (if there are fewer than R larger values than $\alpha_i x_i(t)$, all of those values are removed). The similar removal process is applied to the smaller values. The set of nodes that are removed by node v_i at time t is denoted by $\mathcal{R}_i(t)$. Third, each $v_i \in \mathcal{C}$ updates its value using the following $f_i^D(\cdot)$ in (2):

$$x_i(t+1) = \text{sgn}(\alpha_i) \sum_{v_j \in (\mathcal{N}_i \cup \{v_i\}) \setminus \mathcal{R}_i(t)} w_{ij}(t) \alpha_j x_j^i(t), \quad (13)$$

where $w_{ij}(t)$ are the weights satisfying (iD) $w_{ij}(t) = 0$ if $v_j \notin \mathcal{N}_i \cup \{v_i\}$, (iiD) there exists a constant $\alpha \in (0, 1)$ independent of t , such that $|\alpha_i| w_{ij}(t) \geq \alpha > 0$ for any $v_j \in (\mathcal{N}_i \cup \{v_i\}) \setminus \mathcal{R}_i(t)$, and (iiiD) $\sum_{v_j \in (\mathcal{N}_i \cup \{v_i\}) \setminus \mathcal{R}_i(t)} |\alpha_i| w_{ij}(t) = 1$.

By redefining $M(t) := \max_{v_i \in \mathcal{C}} \alpha_i x_i(t)$ and $m(t) := \min_{v_i \in \mathcal{C}} \alpha_i x_i(t)$, the following result can be established following the similar argument of Theorem 1. We omit the proof due to space limitation.

Theorem 2. Consider the switched multi-agent system (1), (2), (12), (13) over the digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where each cooperative node performs the switched scaled filtering strategy with parameter R . Suppose \mathcal{G} is $2R+1$ -robust and Assumption 1 holds. Then, in the R -locally bounded model with Byzantine nodes, resilient scaled consensus with respect to $(\alpha_1, \dots, \alpha_n)$ is achieved under arbitrary switching.

In the following we further extend the results to solve resilient formation generation problem for switched multi-agent systems in the presence of Byzantine nodes. In formation generation problem, the aim is to design distributed protocols to guarantee that each pair of neighboring agents reach a desired relative position with respect to each other [5, 27]. A certain pattern is thus formed by the agents as a whole. In the context of resilient (scaled) consensus, we introduce the resilient scaled formation generation as follows.

Definition 6. (resilient scaled formation generation) Let $g = (g_1, \dots, g_n) \in \mathbb{R}^n$. The agents in \mathcal{G} are said to achieve resilient scaled formation g with respect to $(\alpha_1, \dots, \alpha_n)$ if $\lim_{t \rightarrow \infty} \alpha_i x_i(t) - \alpha_j x_j(t) = g_i - g_j$ for all $v_i, v_j \in \mathcal{C}$ and all initial conditions $\{x_i(0)\}_{v_i \in \mathcal{V}}$.

It is clear that the agents reach the scaled formation g if there exists a vector $h \in \mathbb{R}^n$ such that for $v_i \in \mathcal{C}$, $\alpha_i x_i(t)$ tends to $g_i + h$ as time goes to infinity. To this end, we design the following formation generation rule for each $v_i \in \mathcal{C}$:

In continuous-time subsystems, $f_i^C(\cdot)$ in (1) is chosen as

$$\dot{x}_i(t) = \text{sgn}(\alpha_i) \sum_{v_j \in (\mathcal{N}_i \cup \{v_i\}) \setminus \mathcal{R}_i(t)} \varphi_{ij}(\alpha_j x_j^i(t) - g_j, \alpha_i x_i(t) - g_i), \quad (14)$$

where the functions φ_{ij} satisfy the same conditions (iC), (iiC), and (iiiC) in the switched scaled filtering strategy with parameter R . In discrete-time subsystems, $f_i^D(\cdot)$ in (2) is chosen as

$$x_i(t+1) = \frac{g_i}{\alpha_i} + \text{sgn}(\alpha_i) \sum_{v_j \in (\mathcal{N}_i \cup \{v_i\}) \setminus \mathcal{R}_i(t)} w_{ij}(t) (\alpha_j x_j^i(t) - g_j), \quad (15)$$

where the weights $w_{ij}(t)$ satisfy the same conditions (iD), (iiD), and (iiiD) in the switched scaled filtering strategy with parameter R . In the formation control problem, we will modify the above three-step switched scaled filtering strategy with parameter R in three places (in continuous-time and discrete-time subsystems, respectively): First, the sorted list is created for $\{\alpha_j x_j^i(t) - g_j\}$ instead of $\{\alpha_j x_j^i(t)\}$. Second, the largest R values that are strictly larger than $\alpha_i x_i(t) - g_i$ instead of $\alpha_i x_i(t)$ in the list are removed. The same modification applies to the smaller values. Third, the update rules (12) and (13) are replaced by (14) and (15), respectively. We shall refer to this modified algorithm as the switched scaled filtering-formation strategy with parameter R .

Corollary 1. Consider the switched multi-agent system (1), (2), (14), (15) over the digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where each cooperative node performs the switched scaled filtering-formation strategy with parameter R . Suppose \mathcal{G} is $2R + 1$ -robust and Assumption 1 holds. Then, in the R -locally bounded model with Byzantine nodes, resilient scaled formation g with respect to $(\alpha_1, \dots, \alpha_n)$ is achieved under arbitrary switching.

Proof. Let $\bar{x}_j^i(t) = x_j^i(t) - g_j/\alpha_j$ and $\bar{x}_j(t) = x_j(t) - g_j/\alpha_j$ for $v_i, v_j \in \mathcal{V}$. Then the update rules (14) and (15) become $\dot{\bar{x}}_i(t) = \text{sgn}(\alpha_i) \sum_{v_j \in (\mathcal{N}_i \cup \{v_i\}) \setminus \mathcal{R}_i(t)} \varphi_{ij}(\alpha_j \bar{x}_j^i(t), \alpha_i \bar{x}_i(t))$ and $\bar{x}_i(t+1) = \text{sgn}(\alpha_i) \sum_{v_j \in (\mathcal{N}_i \cup \{v_i\}) \setminus \mathcal{R}_i(t)} w_{ij}(t) \alpha_j \bar{x}_j^i(t)$, respectively, for $v_i \in \mathcal{C}$.

It follows from Theorem 2 that resilient scaled consensus is achieved for $\{\bar{x}_i(t)\}_{v_i \in \mathcal{C}}$, which is equivalent to having a vector $h \in \mathbb{R}^n$ such that $\lim_{t \rightarrow \infty} \alpha_i x_i(t) = g_i + h$. This completes the proof. \square

Since $g_i + h$ is a constant, our proof is slightly stronger than what is required for resilient scaled formation generation.

Remark 4. It is worth noting that the theoretical results can not be extended to multi-agent systems with multi-dimensional dynamics via existing techniques like the Kronecker product. This is mainly due to (i) the essential nonlinearity involved in continuous-time subsystem (3)

and the generality of the weight functions in discrete-time subsystem (4); and (ii) the difficulty in defining an appropriate vector norm to extend the filtering strategy in a non-trivial way. For instance, in an m -dimensional multi-agent system, a simple generalization of our filtering strategy would require removing a node v_i if any one of the m components of $x_i(t) \in \mathbb{R}^m$ belongs to the R largest (or smallest) values at time t . Our above results remain valid as long as the network \mathcal{G} is $2mR + 1$ -robust. However, this upper bound tends to be conservative.

4. Simulations

In this section, we present a numerical example to illustrate our theoretical results. We consider a 3-robust digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with the node set $\mathcal{V} = \{v_1, \dots, v_6\}$ (see Fig. 1), in which node $v_2 \in \mathcal{B}$ is a Byzantine node. The initial values of the six agents are taken as $x_1(0) = 4$, $x_2(0) = 2$, $x_3(0) = -1$, $x_4(0) = 1$, $x_5(0) = 3$, $x_6(0) = -3$.

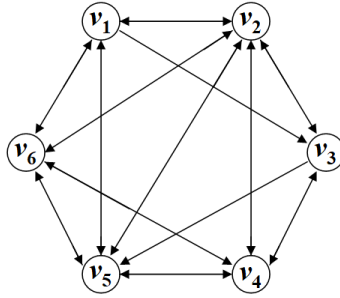


Figure 1: A 3-robust directed graph \mathcal{G} with $n = 6$ nodes.

Since the network is 3-robust, Theorem 1 implies that consensus for system (1)-(4) can be reached in the 1-globally bounded model with Byzantine nodes when the switched filtering strategy with parameter 1 is implemented. The switching law is shown in Fig. 2(a). We assume that each cooperative node $v_i \in \mathcal{C}$ takes the function $\varphi_{ij}(x, y) = 0.1 \cdot (x - y)$ in (3) when the continuous-time subsystem is activated, and the weight $w_{ij}(t) = (|\mathcal{N}_i| + 1 - |\mathcal{R}_i(t)|)^{-1}$ for $v_j \in (\mathcal{N}_i \cup \{v_i\}) \setminus \mathcal{R}_i(t)$ in (4) when the discrete-time subsystem is activated. In Fig. 2(b) we show the trajectories of the agents, where the Byzantine node v_2 updates its value following $\dot{x}_2(t) = 0.02 \cdot x_2(t)$ for continuous-time subsystems and $x_2(t+1) = (x_1(t) + x_3(t) + x_4(t) + x_5(t) + x_6(t))/5 + 0.1 \cdot t$ for discrete-time subsystems. We observe from Fig. 2(b) that the cooperative nodes are able to reach consensus as predicted by Theorem 1.

5. Conclusion

In this letter, we have investigated the resilient consensus problems of switched multi-agent system composed of both continuous-time and discrete-time subsystems. By employing the concept of network robustness and W-MSR procedure, we propose a switched filtering strategy that is able to withstand the compromise of a subset of nodes in directed networks under arbitrary switching rules. Furthermore, resilient scaled consensus problems and resilient scaled formation generation problems for switched multi-agent systems are solved as generalizations. For future work, we will examine resilient consensus of switched multi-agent systems with higher-order

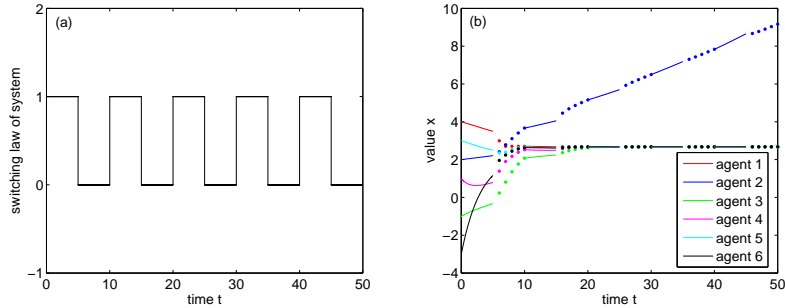


Figure 2: (a) Switching law of the switched multi-agent system (1)-(4). (b) Time evolution of the agents' values, where agent v_2 is a Byzantine node.

and multi-dimensional dynamics, and explore resilient consensus with finite-time convergence rate, etc.

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