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# Phase dynamics of periodic waves leading to the Kadomtsev–Petviashvili equation in $3 + 1$ dimensions

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The Kadomtsev–Petviashvili (KP) equation is a well-known modulation equation normally derived by starting with the trivial state and an appropriate dispersion relation. In this paper, it is shown that the KP equation is also the relevant modulation equation for bifurcation from *periodic travelling waves* when the wave action flux has a critical point. Moreover, the emergent KP equation arises in a universal form, with the coefficients determined by the components of the conservation of wave action. The theory is derived for a general class of partial differential equations generated by a Lagrangian using phase modulation. The theory extends to any space dimension and time, but the emphasis in the paper is on the case of  $3 + 1$ . Motivated by light bullets and quantum vortex dynamics, the theory is illustrated by showing how defocusing NLS in  $3 + 1$  bifurcates to KP in  $3 + 1$  at criticality. The generalization to  $N > 3$  is also discussed.

## 1. Introduction

The Kadomtsev–Petviashvili (KP) equation in  $3 + 1$  can be scaled so that it takes the form

$$(u_t + uu_x + u_{xxx})_x = \pm u_{yy} \pm u_{zz}, \quad (1.1)$$

and in  $N + 1$  with  $N > 3$  one just adds additional second derivative terms for each new space dimension on the right-hand side. The case  $N = 2$  is the classical KP equation first derived in [1]. There has been a

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vast amount of work on 2 + 1 KP and a review can be found in Biondini & Pelinovsky [2]. The 3 + 1 KP has been much less studied. It first appeared in the paper of Kuznetsov & Turitsyn [3], where they study the transverse instability of 2 + 1 lump solitary waves in the 3 + 1 KP equation, showing that they are unstable. Further work, including further detail on the instability of lumps in 2 + 1 → 3 + 1, as well as direct numerical simulation, is reported in Senatorski & Infeld [4] and Infeld *et al.* [5] (see also Infeld & Rowlands [6]). A range of exact solutions of 3 + 1 KP have been discovered (e.g. Ma [7] and references therein).

The interest in this paper is not in solutions or structure of 3 + 1 KP. The contribution of this paper is threefold. We show how and why the KP equation (in any dimension) arises, without recourse to a dispersion relation. The key assumption is that the wave action flux (with the number of components dependent on dimension) has a critical point in wavenumber space. Secondly, it is shown that it is the relevant modulation equation for *periodic travelling waves* with critical wave action flux. Indeed, it would be quite complicated to construct the dispersion relation in general for the linearization about a family of periodic travelling waves, yet the approach based on criticality of the wave action flux is straightforward. Thirdly, we are able to predict the coefficients without recourse to any specific equation, they just follow from the structure of the Lagrangian, and the conservation of wave action. This latter aspect of the theory is reminiscent of Whitham modulation theory (e.g. ch. 14 of [8]), but here the modulation generates dispersion.

We assume that the partial differential equations of interest are generated by a Lagrangian

$$\mathcal{L}(Z) = \iiint \mathcal{L}(Z, Z_t, Z_x, Z_y, Z_z) dx dy dz dt, \quad (1.2)$$

for vector-valued  $Z(x, y, z, t)$ . The Lagrangian (1.2) is for the 3 + 1 case but has obvious extension to higher space dimension.

Suppose there exists a periodic travelling wave solution of the Euler–Lagrange equation

$$Z(x, y, z, t) = \hat{Z}(\theta), \quad \hat{Z}(\theta + 2\pi) = \hat{Z}(\theta), \quad \theta = kx + my + \ell z + \omega t + \theta_0, \quad (1.3)$$

where  $\theta_0$  is an arbitrary phase shift,  $\omega$  is the frequency and  $\mathbf{k} := (k, m, \ell)$  is the wavenumber vector. We assume existence and smoothness of this family of periodic travelling waves.

The form of the emergent KP equation arises by a phase modulation argument. First, the frequency and wavenumber are made explicit in the basic state: replace (1.3) by  $\hat{Z}(\theta, k, m, \ell, \omega)$ , and then introduce a modulation of the phase as well as all parameters

$$Z(x, y, z, t) = \hat{Z}(\theta + \varepsilon\phi, k + \varepsilon^2q, m + \varepsilon^3r, \ell + \varepsilon^3s, \omega + \varepsilon^4\Omega) + \varepsilon^3W(\theta, X, Y, Z, T), \quad (1.4)$$

where  $\phi, q, r, s, \Omega$  are all functions of  $X, Y, Z, T, \varepsilon$ . Although the combination of scales in (1.4) looks strange, it is in fact naturally dictated by the *conservation of waves*, coupled with the scalings (1.11). Recall the classical way to define the local wavenumber and frequency for a given phase

$$\theta_x = k \quad \text{and} \quad \theta_t = \omega,$$

leading to the classical conservation of waves for consistency as

$$k_t - \omega_x = 0. \quad (1.5)$$

By now relating the wavenumber and frequency modulation to derivatives of the phase using the new slow variables, the conservation of waves gives

$$\left. \begin{aligned} q_T &= \Omega_X & q_Y &= r_X & q_Z &= s_X, \\ r_T &= \Omega_Y & r_Z &= s_Y \\ s_T &= \Omega_Z. \end{aligned} \right\} \quad (1.6)$$

and

Each term in (1.4) is scaled so that all the terms in (1.6) are in balance. The conservation of waves (1.6) is a generalization of the 2 + 1 conservation of waves on p. 502–503 of [8].

The expression (1.4) is an ansatz. The strategy is just to substitute (1.4) into the Euler–Lagrange equation associated with (1.2), expand everything in powers of  $\varepsilon$  and equate terms of each order

to zero. We find that the governing equations are satisfied exactly up to fifth order in  $\varepsilon$  if and only if  $q$  satisfies (1.10).

This strategy of ‘phase dynamics’ goes back to Whitham (e.g. ch. 14 of [8]) and the Whitham modulation theory, which was based on a Lagrangian. An inspiration for this work was the reduction theory of Doelman *et al.* [9] which suggested modulating parameters as well as the phase, but that theory involved reduction of reaction–diffusion equations. The theory came full circle in the work of Bridges [10,11] where modulating parameters and new scaling was included in the Lagrangian setting giving a new approach to modulation in the conservative setting. This theory led to a new universal form for the codimension one (only one assumption needed) emergence of the KdV equation. In [11], the KP-II equation in  $2 + 1$  is derived using phase modulation around steady solutions of the water-wave problem. An introduction to modulation in the conservative setting is given in [12].

It follows from the Whitham theory that the Lagrangian has a conservation law for wave action [8, ch. 11 and 14], which we write as

$$A_t + \operatorname{div}(\mathbf{B}) = 0, \quad (1.7)$$

where  $A$  is the wave action, and  $\mathbf{B} := (B, C, D)$  is the wave action flux vector. The  $(A, B, C, D)$  in roman are the components of the conservation law considered as functions of  $Z(x, y, z, t)$ . In the subsequent theory, it is these components *evaluated on the basic state* that are important. Define the wave action evaluated on the basic state (1.3) as

$$\mathcal{A}(\omega, k, m, \ell) = \mathcal{L}_\omega, \quad (1.8)$$

where  $\mathcal{L}$  here is the Lagrangian averaged over the phase of the basic state (1.3). There are similar definitions for  $\mathcal{B}$ ,  $\mathcal{C}$  and  $\mathcal{D}$ . The main result of this paper is that with the assumption

$$\mathcal{B}_k = \mathcal{C}_k = \mathcal{D}_k = 0, \quad (1.9)$$

the perturbation (1.4) of the periodic travelling wave (1.3) satisfies the  $3 + 1$  KP equation

$$((\mathcal{A}_k + \mathcal{B}_\omega)q_T + \mathcal{B}_{kk}qq_X + \mathcal{K}q_{XXX})_X + \mathcal{C}_m q_{YY} + \mathcal{D}_\ell q_{ZZ} = 0 \quad (1.10)$$

to leading order in  $\varepsilon$  (where  $\varepsilon$  is a small parameter to be defined). In this equation,  $T$ ,  $X$ ,  $Y$  and  $Z$  are slow time and space scales

$$X = \varepsilon x, \quad Y = \varepsilon^2 y, \quad Z = \varepsilon^2 z \quad \text{and} \quad T = \varepsilon^3 t. \quad (1.11)$$

In order to generate further second-order derivatives on the right-hand side of (1.10) the additional space dimensions would also have slow versions of order  $\varepsilon^2$ . In (1.10), the dependent variable  $q(X, Y, Z, T, \varepsilon)$  arises as a modulation of the  $x$ -direction wavenumber  $k$ . The assumption  $\mathcal{B}_k = 0$  generates the nonlinearity  $qq_X$ , and the assumptions  $\mathcal{C}_k = \mathcal{D}_k = 0$  assure that the right-hand side of (1.10) contains dispersion terms consisting of second derivatives.

The remarkable feature of the  $3 + 1$  KP in (1.10) is that the coefficients other than  $\mathcal{K}$  are determined by derivatives of the components of the wave action conservation law, evaluated on the basic state. In particular, the coefficients of transverse dispersion  $\mathcal{C}_m$  and  $\mathcal{D}_\ell$  are completely determined by properties of the family of periodic travelling waves. The equation is universal in the sense that it does not rely on properties of a particular equation, it just follows from the Lagrangian structure and the conservation of wave action. The parameter  $\mathcal{K}$  is the odd one. It is also a dispersion parameter but it arises due to a symplectic Jordan chain theory argument.

The strategy of this paper—introduce an ansatz, substitute into the Euler–Lagrange equation, derive exact equations up to fifth order and show that the coefficients are determined by a conservation law—is similar to [10] and so we will be brief, highlighting those features that are new and different. Indeed, the first three terms in (1.10) are the same as in [10]. The two key new features are the form of the transverse dispersion, and the fact that the reduction in the  $N + 1$  case is codimension  $N$  (e.g. (1.9)). Although we will show that the codimension can be reduced when the system has a transverse reflection symmetry.

Our principal example is the reduction of the 3 + 1 defocusing nonlinear Schrödinger (NLS) equation to the 3 + 1 KP equation. The defocusing NLS in 3 + 1 has solitary wave solutions that are known as bullets due to their localized form in three space dimensions and they have attracted recent interest [13]. Although normally found in NLS with variable coefficients, the 3 + 1 KP has localized solutions that are similar to bullets [5], capturing a reduction of the three-dimensional localized solutions in defocusing NLS [14]. Another motivation for studying 3 + 1 NLS is quantum vortices (cf. Kerr [15] and references therein). In §8, we show how the theory in this paper gives immediately the coefficients in the 3 + 1 KP derived from 3 + 1 NLS. This reduction generalizes the reduction of NLS in 2 + 1 to KP-I (e.g. [16–18]).

An outline of the paper is as follows. The Lagrangian set-up, including structure of the Lagrangian, averaging, linearization and the conservation of wave action are introduced in §§2 and 3. Sections 4–6 give details of the modulation expansion and ordering of terms. When the system has a transverse reflection symmetry, the codimension of the emergence of KP is reduced by each such symmetry. The argument behind this is sketched in §7. The calculations giving rise to the reduction from 3 + 1 NLS to 3 + 1 KP are given in §8. In §9, the extension to any space dimension  $N > 3$  is outlined.

## 2. From Lagrangian to multisymplectic Hamiltonian

It is easier to proceed with the theory when the Lagrangian has structure. The strategy is to transform the Lagrangian density to a multisymplectic Hamiltonian density [19,20]. In this formulation, the conservation of wave action is given a geometric formulation [21] with a direct link to the equations.

The transformation from Lagrangian to multisymplectic Hamiltonian is effectively a multiple Legendre transform. Start with the *Lagrangian* formulation for some PDE

$$\mathcal{L}^L(U) = \iiint \mathcal{L}(U_t, U_x, U_y, U_z, U) \, dx \, dy \, dz \, dt, \quad (2.1)$$

where  $U(x, y, z, t)$  is in general vector valued. Legendre transform  $V = \delta L / \delta U_t$ , giving a *Hamiltonian formulation*

$$\mathcal{L}^H(W) = \iiint \left[ \frac{1}{2} \langle \mathbf{M}W_t, W \rangle - H(W_x, W_y, W_z, W) \right] \, dx \, dy \, dz \, dt, \quad (2.2)$$

with new coordinates  $W = (U, V)$ , and  $\langle \cdot, \cdot \rangle$  an appropriate inner product, with  $\mathbf{M}$  and  $H$  defined by Legendre transform. The density is still the same Lagrangian density with new coordinates. The advantage is that it has been split into two parts: a Hamiltonian function  $H(W_x, W_y, W_z, W)$  which is scalar valued, and a part defined by a symplectic operator  $\mathbf{M}$ , which for the purposes of this paper can be taken to be a constant skew-symmetric matrix.

Now continue to Legendre transform the Hamiltonian function in each space direction, resulting in a *multisymplectic Hamiltonian* formulation

$$\mathcal{L}(Z) = \iiint \left[ \frac{1}{2} \langle \mathbf{M}Z_t, Z \rangle + \frac{1}{2} \langle \mathbf{J}Z_x, Z \rangle + \frac{1}{2} \langle \mathbf{K}Z_y, Z \rangle + \frac{1}{2} \langle \mathbf{P}Z_z, Z \rangle - S(Z) \right] \, dx \, dy \, dz \, dt, \quad (2.3)$$

with new coordinates  $Z(x, y, z, t)$ , and  $\langle \cdot, \cdot \rangle$  an appropriate inner product. The density is again the same Lagrangian density in terms of the new coordinates, but now it is split into  $N + 2$  parts, where  $N$  is the space dimension: a new Hamiltonian function  $S(Z)$  which does not contain any derivatives with respect to  $t, x, y, z$  and  $N + 1$  symplectic structures represented by the skew-symmetric matrices  $\mathbf{M}, \mathbf{J}, \mathbf{K}, \mathbf{P}$ . The principal advantage of the multisymplectic structure is that the symplectic structures appear both in the equations and in the conservation of wave action, giving an explicit connection for the modulation theory.

The above sequence of Legendre transforms is schematic, as in general non-degeneracy conditions are required, and each PDE has to be treated with care. An example of the above sequence of Legendre transforms is given in §8.

### 3. Euler–Lagrange equations and modulation

The starting point for the theory is the Euler–Lagrange equation associated with the Lagrangian (2.3)

$$\mathbf{M}Z_t + \mathbf{J}Z_x + \mathbf{K}Z_y + \mathbf{P}Z_z = \nabla S(Z), \quad Z \in \mathbb{R}^n, \quad (3.1)$$

for some  $n \geq 4$ . Here,  $S: \mathbb{R}^n \rightarrow \mathbb{R}$  is the Hamiltonian function,  $\mathbf{M}, \mathbf{J}, \mathbf{K}, \mathbf{P}$  are constant skew symmetric matrices,

$$\mathbf{M}^T = -\mathbf{M}, \quad \mathbf{J}^T = -\mathbf{J}, \quad \mathbf{K}^T = -\mathbf{K}, \quad \mathbf{P}^T = -\mathbf{P}.$$

We also assume the existence of a family of periodic travelling wave solutions parametrized by their wavenumbers  $k, m, \ell$  and frequency  $\omega$ ,

$$Z(x, y, z, t) = \hat{Z}(\theta, k, m, \ell, \omega), \quad \theta = kx + my + \ell z + \omega t + \theta_0$$

for some constant phase shift  $\theta_0$ . Periodicity requires  $\hat{Z}(\theta + 2\pi, k, m, \ell, \omega) = \hat{Z}(\theta, k, m, \ell, \omega)$ . The travelling wave solution satisfies

$$(\omega \mathbf{M} + k \mathbf{J} + m \mathbf{K} + \ell \mathbf{P}) \hat{Z}_\theta = \nabla S(\hat{Z}), \quad (3.2)$$

where the variable subscript denotes differentiation.

The modulation ansatz is given in (1.4). The strategy is to substitute this modulation ansatz into the governing equations and equate like powers of  $\varepsilon$  to zero. First, preliminary results on the derivatives of the basic state and their connection with wave action conservation are established.

#### (a) Averaging the Lagrangian and wave action

To get the components of the conservation law for wave action, average (2.3), evaluated on the family of travelling waves, over  $\theta$ ,

$$\mathcal{L}(\omega, k, m, \ell) = \frac{1}{2\pi} \int_0^{2\pi} \left[ \frac{\omega}{2} \langle \mathbf{M} \hat{Z}_\theta, \hat{Z} \rangle + \frac{k}{2} \langle \mathbf{J} \hat{Z}_\theta, \hat{Z} \rangle + \frac{m}{2} \langle \mathbf{K} \hat{Z}_\theta, \hat{Z} \rangle + \frac{\ell}{2} \langle \mathbf{P} \hat{Z}_\theta, \hat{Z} \rangle - S(\hat{Z}) \right] d\theta,$$

and differentiate with respect to  $\omega, k, m, \ell$ , giving

$$\left. \begin{aligned} \mathcal{A}(\omega, k, m, \ell) &= \mathcal{L}_\omega = \frac{1}{2} \langle \mathbf{M} \hat{Z}_\theta, \hat{Z} \rangle, \\ \mathcal{B}(\omega, k, m, \ell) &= \mathcal{L}_k = \frac{1}{2} \langle \mathbf{J} \hat{Z}_\theta, \hat{Z} \rangle, \\ \mathcal{C}(\omega, k, m, \ell) &= \mathcal{L}_m = \frac{1}{2} \langle \mathbf{K} \hat{Z}_\theta, \hat{Z} \rangle \\ \mathcal{D}(\omega, k, m, \ell) &= \mathcal{L}_\ell = \frac{1}{2} \langle \mathbf{P} \hat{Z}_\theta, \hat{Z} \rangle. \end{aligned} \right\} \quad (3.3)$$

and

where  $\langle \cdot, \cdot \rangle$  is the inner product averaged over  $\theta$ ,

$$\langle \langle U, V \rangle \rangle := \frac{1}{2\pi} \int_0^{2\pi} \langle U, V \rangle d\theta. \quad (3.4)$$

Note that the structure matrices  $\mathbf{M}, \mathbf{J}, \mathbf{K}, \mathbf{P}$  appear both in the Euler–Lagrange equations, (3.1) and in the components of the conservation law (3.3).

The derivatives in (1.9) are

$$\mathcal{B}_k = \langle \langle \mathbf{J} \hat{Z}_\theta, \hat{Z}_k \rangle \rangle, \quad \mathcal{C}_k = \langle \langle \mathbf{K} \hat{Z}_\theta, \hat{Z}_k \rangle \rangle \quad \text{and} \quad \mathcal{D}_k = \langle \langle \mathbf{P} \hat{Z}_\theta, \hat{Z}_k \rangle \rangle. \quad (3.5)$$

The coefficient of the time derivative is

$$\mathcal{A}_k = \langle \langle \mathbf{M} \hat{Z}_\theta, \hat{Z}_k \rangle \rangle = \langle \langle \mathbf{J} \hat{Z}_\theta, \hat{Z}_\omega \rangle \rangle = \mathcal{B}_\omega, \quad (3.6)$$

with equality following from  $\mathcal{A}_k = \mathcal{L}_{\omega k} = \mathcal{L}_{k\omega} = \mathcal{B}_\omega$ . Similar cross derivatives exist for the other components of wave action, which are useful for simplifying the modulation theory.

We will also need the second  $k$  derivative of  $\mathcal{B}$ ,

$$\mathcal{B}_{kk} = \langle\langle \mathbf{J}\hat{Z}_\theta, \hat{Z}_{kk} \rangle\rangle + \langle\langle \mathbf{J}\hat{Z}_{\theta k}, \hat{Z}_k \rangle\rangle, \quad (3.7)$$

as well as the derivatives

$$\mathcal{C}_m = \langle\langle \mathbf{K}\hat{Z}_\theta, \hat{Z}_m \rangle\rangle \quad \text{and} \quad \mathcal{D}_\ell = \langle\langle \mathbf{P}\hat{Z}_\theta, \hat{Z}_\ell \rangle\rangle. \quad (3.8)$$

## (b) Linearization about the periodic basic state

Define the linear operator

$$\mathbf{L}f = \left[ D^2S(\hat{Z}) - k\mathbf{J}\frac{d}{d\theta} - \ell\mathbf{K}\frac{d}{d\theta} - m\mathbf{P}\frac{d}{d\theta} - \omega\mathbf{M}\frac{d}{d\theta} \right] f, \quad (3.9)$$

obtained by linearizing (3.2). Then differentiating (3.2) with respect to  $\theta$  and  $k$ ,

$$\begin{aligned} D^2S(\hat{Z})\hat{Z}_\theta &= k\mathbf{J}\hat{Z}_{\theta\theta} + \omega\mathbf{M}\hat{Z}_{\theta\theta}, \\ D^2S(\hat{Z})\hat{Z}_k &= k\mathbf{J}\hat{Z}_{\theta k} + \omega\mathbf{M}\hat{Z}_{\theta k} + \mathbf{J}\hat{Z}_\theta \end{aligned}$$

or

$$\mathbf{L}\hat{Z}_\theta = 0 \quad \text{and} \quad \mathbf{L}\hat{Z}_k = \mathbf{J}\hat{Z}_\theta. \quad (3.10)$$

Other equations of interest in the modulation theory are the differentiation of (3.2) with respect to  $\omega, m, \ell$  which give

$$\mathbf{L}\hat{Z}_\omega = \mathbf{M}\hat{Z}_\theta, \quad \mathbf{L}\hat{Z}_\ell = \mathbf{K}\hat{Z}_\theta \quad \text{and} \quad \mathbf{L}\hat{Z}_m = \mathbf{P}\hat{Z}_\theta. \quad (3.11)$$

The first equation of (3.10) shows that  $\hat{Z}_\theta$  is in the Kernel of  $\mathbf{L}$ . It is natural to assume that the kernel is no larger. Hence assume

$$\text{Kernel}(\mathbf{L}) = \text{span}\{\hat{Z}_\theta\}. \quad (3.12)$$

The second equation of (3.10) shows that there is a non-trivial Jordan chain associated with the zero eigenvalue of  $\mathbf{L}$  with geometric eigenvector  $\hat{Z}_\theta$ . This Jordan chain is discussed in §5.

For inhomogeneous equations that arise in the modulation theory and the Jordan chain theory, a solvability condition will be needed. With the assumption (3.12) and the symmetry of  $\mathbf{L}$ , the solvability condition for the inhomogeneous equation  $\mathbf{L}W = F$  is

$$\mathbf{L}W = F \quad \text{is solvable if and only if} \quad \langle\langle \hat{Z}_\theta, F \rangle\rangle = 0. \quad (3.13)$$

## 4. Details of the modulation expansion

The aim is to expand the modulation ansatz (1.4) in powers of  $\varepsilon$ , transform the derivatives using the chain rule, and then solve the equations at each order in  $\varepsilon$ . The small parameter  $\varepsilon$  is a measure of the distance in  $k$  space from criticality. Let  $k_0$  be a value of  $k$  satisfying  $\mathcal{B}_k = 0$  then  $k - k_0 = \varepsilon^2 q$  with  $q$  of order one. Taylor expanding the modulation of the basic state, we can write

$$\hat{Z}(\theta + \varepsilon\phi, k + \varepsilon^2 q, m + \varepsilon^3 r, \ell + \varepsilon^3 s, \omega + \varepsilon^4 \Omega) = \sum_{n=0}^5 \varepsilon^n Z_n + \mathcal{O}(\varepsilon^6)$$

with

$$\begin{aligned} Z_0 &= \hat{Z}(\theta, k, m, \omega), \quad Z_1 = \phi\hat{Z}_\theta, \quad Z_2 = \frac{1}{2}\phi^2\hat{Z}_{\theta\theta} + q\hat{Z}_k, \\ Z_3 &= \frac{1}{6}\phi^3\hat{Z}_{\theta\theta\theta} + q\phi\hat{Z}_{\theta k} + r\hat{Z}_m + s\hat{Z}_\ell, \\ Z_4 &= \frac{1}{24}\phi^4\hat{Z}_{\theta\theta\theta\theta} + \frac{1}{2}q\phi^2\hat{Z}_{\theta\theta k} + \frac{1}{2}q^2\hat{Z}_{kk} + \phi r\hat{Z}_{\theta m} + \phi s\hat{Z}_{\theta\ell} + \Omega\hat{Z}_\omega, \\ Z_5 &= \frac{1}{120}\phi^5\hat{Z}_{\theta\theta\theta\theta\theta} + \frac{1}{6}q\phi^3\hat{Z}_{\theta\theta\theta k} + \frac{1}{2}q^2\phi\hat{Z}_{\theta kk} + \frac{1}{2}\phi^2 r\hat{Z}_{\theta\theta m} + \frac{1}{2}\phi^2 s\hat{Z}_{\theta\theta\ell} \\ &\quad + qr\hat{Z}_{km} + qs\hat{Z}_{k\ell} + \Omega\phi\hat{Z}_{\theta\omega}, \end{aligned}$$

with each term in the expansion evaluated at  $\theta, k, m, \ell$ . Expand the remainder term  $W$  as well

$$W = W_0 + \varepsilon W_1 + \varepsilon^2 W_2 + \mathcal{O}(\varepsilon^3).$$

The full expansions of the terms appearing in the governing equations are lengthy and are therefore just a summary of the key steps is presented.

The zeroth-order equation is just the equation for the basic state recovering (3.2). The first-order equation gives  $\phi \mathbf{L} \hat{Z}_\theta$  which is satisfied exactly due to (3.10). The second-order equation just recovers the definition  $q = \phi_X$ .

### (a) Third-order terms

At third order, terms proportional to  $\phi^3$  and  $q\phi$  can be shown to vanish identically. For example, the terms proportional to  $\phi^3$  are

$$\frac{1}{6} \phi^3 (D^4 S(\hat{Z})(\hat{Z}_\theta, \hat{Z}_\theta, \hat{Z}_\theta) + 3D^3 S(\hat{Z})(\hat{Z}_\theta, \hat{Z}_{\theta\theta}) + \mathcal{L}_0 \hat{Z}_{\theta\theta\theta}),$$

which is identically zero, and can be seen by differentiation of (3.2) three times with respect to  $\theta$ . The  $q\phi$  terms vanish under a similar argument. This leaves

$$(r - \phi_Y) \mathbf{L} \hat{Z}_m + (s - \phi_Z) \mathbf{L} \hat{Z}_\ell + \mathbf{L} W_0 - q_X \mathbf{J} \hat{Z}_k = 0. \quad (4.1)$$

The first two terms vanish by construction since  $r = \phi_Y$  and  $s = \phi_Z$ . Then this system is considered solvable if

$$\langle\langle \hat{Z}_\theta, \mathbf{J} \hat{Z}_k \rangle\rangle = -\langle\langle \mathbf{J} \hat{Z}_\theta, \hat{Z}_k \rangle\rangle = -\mathcal{B}_k = 0, \quad (4.2)$$

and so we require that  $\mathcal{B}$  is extremal in  $k$  in order to continue with the asymptotic analysis. This solvability condition confirms the first necessary condition in (1.9). The solution for  $W_0$  is then

$$W_0 = \alpha(X, Y, Z, T) \hat{Z}_\theta + q_X \xi_3, \quad (4.3)$$

for some arbitrary function  $\alpha$ , and where  $\xi_3$  is defined through the relation

$$\mathbf{L} \xi_3 = \mathbf{J} \hat{Z}_k. \quad (4.4)$$

This equation is solvable under the condition  $\mathcal{B}_k = 0$ .

## 5. Interlude: Jordan chains

A Jordan chain of length  $J$ ,  $\{\xi_1, \dots, \xi_J\}$ , for a zero eigenvalue in the symplectic setting is defined by

$$\mathbf{L} \xi_1 = 0 \quad \text{and} \quad \mathbf{L} \xi_i = \mathbf{J} \xi_{i-1}, \quad i = 2, \dots, J. \quad (5.1)$$

When  $\mathbf{J}$  is invertible then this chain is a classical Jordan chain. However, in this case,  $\mathbf{J}$  may not be invertible. Hence we include the assumption

$$\xi_i \notin \text{Ker}(\mathbf{J}), \quad i = 1, \dots, J, \quad (5.2)$$

which appears to be satisfied in examples.

Here we are interested in the Jordan chain associated with the geometric eigenvector  $\xi_1 = \hat{Z}_\theta$ . As shown in (3.10), the Jordan chain associated with  $\hat{Z}_\theta$  has length at least two since  $\xi_2 = \hat{Z}_k$ . In fact, it has length at least three due to (4.4). It has length four if  $\mathbf{L} \xi_4 = \mathbf{J} \xi_3$  and this equation is

solvable if and only if

$$\langle\langle \hat{Z}_\theta, \mathbf{J}\xi_3 \rangle\rangle = 0. \quad (5.3)$$

However,

$$\begin{aligned} \langle\langle \hat{Z}_\theta, \mathbf{J}\xi_3 \rangle\rangle &= -\langle\langle \mathbf{J}\hat{Z}_\theta, \xi_3 \rangle\rangle = -\langle\langle \mathbf{L}\hat{Z}_k, \xi_3 \rangle\rangle \\ &= -\langle\langle \hat{Z}_k, \mathbf{L}\xi_3 \rangle\rangle = -\langle\langle \hat{Z}_k, \mathbf{J}\hat{Z}_k \rangle\rangle = 0, \end{aligned} \quad (5.4)$$

with the last equality due to skew symmetry of  $\mathbf{J}$ . Hence the Jordan chain has length at least four. There is no fifth element if

$$\langle\langle \xi_1, \mathbf{J}\xi_4 \rangle\rangle \neq 0. \quad (5.5)$$

Assume this condition is satisfied and define

$$\mathcal{H} = \langle\langle \mathbf{J}\xi_1, \xi_4 \rangle\rangle \neq 0. \quad (5.6)$$

It is precisely this coefficient that arises in the modulation theory to give the coefficient of dispersion in the  $x$ -direction.

## 6. Terms of order four and five in the expansion

At fourth-order, the equation simplifies to

$$\begin{aligned} (\Omega - \phi_T)\mathbf{L}\hat{Z}_\omega + \mathbf{L}(W_1 - \alpha_X\hat{Z}_k - \alpha\phi\hat{Z}_{\theta\theta}) - q_Y(\mathbf{J}\hat{Z}_m + \mathbf{K}\hat{Z}_k) \\ - q_Z(\mathbf{J}\hat{Z}_\ell + \mathbf{P}\hat{Z}_k) + q_X\phi(D^3S(\hat{Z})(\hat{Z}_\theta, \xi_3) - \mathbf{J}\hat{Z}_{\theta k}) - q_{XX}\mathbf{J}\xi_3 = 0, \end{aligned} \quad (6.1)$$

where simplifications have been introduced to account for terms that vanish identically, and the identities,  $q_Y = r_X$  and  $q_Z = s_X$ , from the conservation of waves have been used.

Note that the first term vanishes if  $\Omega = \phi_T$ , a similar enforcement to the previous orders. The term prefactored by  $q_X\phi$  can be shown to be the result of  $\mathbf{L}(\xi_3)_\theta$  since if we differentiate its defining equation (4.4) with respect to  $\theta$ ,

$$\begin{aligned} D^2S(\hat{Z})(\xi_3)_\theta + D^3S(\hat{Z})(\hat{Z}_\theta, \xi_3) - (\omega\mathbf{M} + k\mathbf{J} + m\mathbf{K})(\xi_3)_\theta &= \mathbf{J}\hat{Z}_{\theta k} \\ \Rightarrow \mathbf{L}(\xi_3)_\theta &= \mathbf{J}\hat{Z}_{\theta k} - D^3S(\hat{Z})(\hat{Z}_\theta, \xi_3). \end{aligned}$$

What can we do about that  $q_Y$  term? Checking solvability:

$$\begin{aligned} \langle\langle \hat{Z}_\theta, \mathbf{J}\hat{Z}_m + \mathbf{K}\hat{Z}_k \rangle\rangle &= -\langle\langle \mathbf{J}\hat{Z}_\theta, \hat{Z}_m \rangle\rangle + \langle\langle \hat{Z}_\theta, \mathbf{K}\hat{Z}_k \rangle\rangle \\ &= -\langle\langle \mathbf{L}\hat{Z}_k, \hat{Z}_m \rangle\rangle + \langle\langle \hat{Z}_\theta, \mathbf{K}\hat{Z}_k \rangle\rangle = -\langle\langle \hat{Z}_k, \mathbf{L}\hat{Z}_m \rangle\rangle + \langle\langle \hat{Z}_\theta, \mathbf{K}\hat{Z}_k \rangle\rangle \\ &= -\langle\langle \hat{Z}_k, \mathbf{K}\hat{Z}_\theta \rangle\rangle + \langle\langle \hat{Z}_\theta, \mathbf{K}\hat{Z}_k \rangle\rangle = 2\langle\langle \hat{Z}_\theta, \mathbf{K}\hat{Z}_k \rangle\rangle = -2\mathcal{C}_k. \end{aligned}$$

Thus, it is only solvable if

$$\mathcal{C}_k = 0. \quad (6.2)$$

The second component of the wave action flux  $\mathcal{C}$  needs to be extremal with respect to  $k$  to continue. Similarly, to solve for the  $q_Z$  term, we consider its inner product:

$$\begin{aligned} \langle\langle \hat{Z}_\theta, \mathbf{J}\hat{Z}_\ell + \mathbf{P}\hat{Z}_k \rangle\rangle &= -\langle\langle \mathbf{J}\hat{Z}_\theta, \hat{Z}_\ell \rangle\rangle - \langle\langle \mathbf{P}\hat{Z}_\theta, \hat{Z}_k \rangle\rangle \\ &= -\langle\langle \mathbf{L}\hat{Z}_k, \hat{Z}_\ell \rangle\rangle - \langle\langle \mathbf{P}\hat{Z}_\theta, \hat{Z}_k \rangle\rangle = -\langle\langle \hat{Z}_k, \mathbf{L}\hat{Z}_\ell \rangle\rangle - \langle\langle \mathbf{P}\hat{Z}_\theta, \hat{Z}_k \rangle\rangle \\ &= -\langle\langle \hat{Z}_k, \mathbf{P}\hat{Z}_\theta \rangle\rangle - \langle\langle \mathbf{P}\hat{Z}_\theta, \hat{Z}_k \rangle\rangle = -2\langle\langle \mathbf{P}\hat{Z}_\theta, \hat{Z}_k \rangle\rangle = -2\mathcal{D}_k. \end{aligned}$$

So we also need  $k$  extremality in  $\mathcal{D}$  in order to continue. Overall, the resulting solution for  $W_1$  at this order is

$$W_1 = \alpha_X\hat{Z}_k + \alpha\phi\hat{Z}_{\theta\theta} + \beta(X, Y, T)\hat{Z}_\theta + q_X\phi(\xi_3)_\theta + q_Y\eta + q_Z\zeta + q_{XX}\xi_4, \quad (6.3)$$

with

$$\mathbf{L}\eta = \mathbf{J}\hat{Z}_m + \mathbf{K}\hat{Z}_k, \quad \mathbf{L}\zeta = \mathbf{J}\hat{Z}_\ell + \mathbf{P}\hat{Z}_k, \quad (6.4)$$



which are solvable when  $\mathcal{C}_k = \mathcal{D}_k = 0$ . By cross differentiation of the Lagrangian, these two conditions are equivalent to imposing that  $\mathcal{B}_m = \mathcal{B}_l = 0$  and naturally leads to equivalent conditions to (1.9) for the KP to emerge.

### (a) Fifth-order terms

After a few simplifications at this final order, we have the equation

$$\begin{aligned} \mathbf{L}(W_2 - \alpha_{XX}\xi_3 - (\alpha_X\phi + \alpha q)\hat{Z}_{\theta k} - \alpha_Y\hat{Z}_m - \frac{1}{2}\phi^2\alpha\hat{Z}_{\theta\theta\theta} - \beta_X\hat{Z}_k \\ - \beta\phi\hat{Z}_{\theta\theta} - \frac{1}{2}q_X\phi^2(\xi_3)_{\theta\theta} - q_Y\phi(\eta)_{\theta} - q_{XX}(\xi_4)_{\theta}) - \alpha_Z\hat{Z}_{\ell} - q_Z(\zeta)_{\theta}) \\ - q_T(\mathbf{M}\hat{Z}_k + \mathbf{J}\hat{Z}_{\omega}) - qq_X(\mathbf{J}\hat{Z}_{kk} + \mathbf{J}(\xi_3)_{\theta}) - D^3S(\hat{Z})(\hat{Z}_k, \xi_3) \\ - r_Y\mathbf{K}\hat{Z}_m - p_Z\mathbf{P}\hat{Z}_{\ell} - q_{XXX}\mathbf{J}\xi_4 = 0. \end{aligned} \quad (6.5)$$

Terms that are now acted on by the linear operator can be shown to simplify this way by similar reasoning used at the previous orders. We now take the inner product of the above. The first term outside of the linear operator yields

$$\langle\langle \hat{Z}_{\theta}, \mathbf{M}\hat{Z}_k + \mathbf{J}\hat{Z}_{\omega} \rangle\rangle = \langle\langle \hat{Z}_{\theta}, \mathbf{M}\hat{Z}_k \rangle\rangle + \langle\langle \hat{Z}_{\theta}, \mathbf{J}\hat{Z}_{\omega} \rangle\rangle = -\langle\langle \mathbf{M}\hat{Z}_{\theta}, \hat{Z}_k \rangle\rangle - \langle\langle \mathbf{J}\hat{Z}_{\theta}, \hat{Z}_{\omega} \rangle\rangle = -\mathcal{A}_k - \mathcal{B}_{\omega} = -2\mathcal{A}_k. \quad (6.6)$$

The last one is simply

$$\langle\langle \hat{Z}_{\theta}, \mathbf{J}\xi_4 \rangle\rangle = -\mathcal{H}.$$

The  $r_Y$  coefficient is also simple to evaluate and gives the result

$$\langle\langle \hat{Z}_{\theta}, \mathbf{K}\hat{Z}_m \rangle\rangle = -\langle\langle \mathbf{K}\hat{Z}_{\theta}, \hat{Z}_m \rangle\rangle = -\mathcal{C}_m. \quad (6.7)$$

For the hydrodynamic term, we start with the definition of  $\mathcal{B}_{kk}$  to show that

$$\begin{aligned} \mathcal{B}_{kk} &= \langle\langle \mathbf{J}\hat{Z}_{\theta}, \hat{Z}_{kk} \rangle\rangle + \langle\langle \mathbf{J}\hat{Z}_{\theta k}, \hat{Z}_k \rangle\rangle = \langle\langle \mathbf{J}\hat{Z}_{\theta}, \hat{Z}_{kk} \rangle\rangle - \langle\langle \hat{Z}_{\theta k}, \mathbf{J}\hat{Z}_k \rangle\rangle \\ &= \langle\langle \mathbf{J}\hat{Z}_{\theta}, \hat{Z}_{kk} \rangle\rangle - \langle\langle \hat{Z}_{\theta k}, \mathbf{L}\xi_3 \rangle\rangle = \langle\langle \mathbf{J}\hat{Z}_{\theta}, \hat{Z}_{kk} \rangle\rangle - \langle\langle \mathbf{L}\hat{Z}_{\theta k}, \xi_3 \rangle\rangle \\ &= \langle\langle \mathbf{J}\hat{Z}_{\theta}, \hat{Z}_{kk} \rangle\rangle - \langle\langle \mathbf{J}\hat{Z}_{\theta\theta} - D^3S(\hat{Z})(\hat{Z}_k, \hat{Z}_{\theta}), \xi_3 \rangle\rangle \\ &= -\langle\langle \hat{Z}_{\theta}, \mathbf{J}\hat{Z}_{kk} \rangle\rangle - \langle\langle \mathbf{J}\hat{Z}_{\theta\theta}, \xi_3 \rangle\rangle + \langle\langle D^3S(\hat{Z})(\hat{Z}_k, \hat{Z}_{\theta}), \xi_3 \rangle\rangle \\ &= -\langle\langle \hat{Z}_{\theta}, \mathbf{J}\hat{Z}_{kk} \rangle\rangle - \langle\langle \hat{Z}_{\theta}, \mathbf{J}(\xi_3)_{\theta} \rangle\rangle + \langle\langle \hat{Z}_{\theta}, D^3S(\hat{Z})(\hat{Z}_k, \xi_3) \rangle\rangle. \end{aligned} \quad (6.8)$$

This reveals that the hydrodynamic term is the negative curvature of  $\mathcal{B}$  with respect to the first wavenumber. The last to compute is the coefficient of the  $p_Z$  term. Via calculation:

$$\langle\langle \hat{Z}_{\theta}, \mathbf{P}\hat{Z}_{\ell} \rangle\rangle = -\langle\langle \mathbf{P}\hat{Z}_{\theta}, \hat{Z}_{\ell} \rangle\rangle = -\mathcal{D}_{\ell}. \quad (6.9)$$

Therefore, solvability requires that

$$2\mathcal{A}_k q_T + \mathcal{B}_{kk} q q_X + \mathcal{H} q_{XXX} + \mathcal{C}_m r_Y + \mathcal{D}_{\ell} p_Z = 0. \quad (6.10)$$

Taking the  $X$  derivative of this equation gives (1.10) and thus the 3 + 1 KP equation governs the dynamics of the perturbation at fifth order.

The  $qq_X$  bracket in the above appears to be the  $k$  derivative of the  $\xi_3$  equation,

$$\begin{aligned} D^3S(\hat{Z})(\hat{Z}_k, \xi_3) + D^2S(\hat{Z})(\xi_3)_k - (\omega\mathbf{M} + k\mathbf{J} + m\mathbf{K})(\xi_3)_{\theta k} - \mathbf{J}(\xi_3)_{\theta} = \mathbf{J}\hat{Z}_{kk} \\ \Rightarrow \mathbf{L}(\xi_3)_k = \mathbf{J}\hat{Z}_{kk} + \mathbf{J}(\xi_3)_{\theta} - D^3S(\hat{Z})(\hat{Z}_k, \xi_3). \end{aligned} \quad (6.11)$$

This would appear to imply that the coefficient of the hydrodynamic term,  $qq_X$ , vanishes identically. However, the  $k$  derivative for this vector does not necessarily exist:  $\xi_3$  only exists for specific values of the wavenumber, defined by  $\mathcal{B}_k = 0$ , and so  $\xi_3$  is not necessarily differentiable and so the coefficient of  $qq_X$  is, generically, non-zero.

## (b) Reduction to the 2 + 1 case

The most widely studied case of the KP equation is the 2 + 1 case [2]. The theory here reduces immediately to that case by restricting the original PDE to have coordinates  $(x, y, t)$  only. The 3 + 1 KP reduces to this case by neglecting  $Z$ -dependence giving

$$(2\mathcal{A}_k q_T + \mathcal{B}_{kk} q q_X + \mathcal{K} q_{XXX})_X + \mathcal{C}_m q_{YY} = 0.$$

Even in this special case the theory points to new results. The typical derivation of the KP equation in 2 + 1 is relative to the trivial state, and the theory here shows how the 2 + 1 KP equation can arise relative to a non-trivial periodic travelling wave. Hence it points towards new applications of the KP equation.

One of the most well-known contexts for the appearance of the 2 + 1 KP equation as a model equation is in the theory of water waves. A special case of the modulation approach was used in Bridges [11] to give a new derivation of the KP-II equation in shallow water, and showed the connection between the coefficients and the properties of classical uniform flows. The theory of this paper suggests that the KP equation may also appear as a modulation equation in water waves in the perturbation about non-trivial periodic travelling waves.

## 7. Implications of a transverse reflection symmetry

One of the curiosities of the emergence of the KP equation is that it is codimension  $N$  where  $N - 1$  is the number of transverse space directions (meaning that  $N$  conditions (1.9) are necessary for emergence). On the other hand, the KdV equation is codimension 1, requiring only the condition  $\mathcal{B}_k = 0$  [10], and the KP equation should be just as prevalent, that is, also codimension 1.

This contradiction is rectified by noting that when the governing equations have a transverse reflection symmetry in the  $y$ -direction then the condition  $\mathcal{C}_m = 0$  is automatically satisfied when  $m = 0$ . Similarly when there is a transverse reflection symmetry in the  $z$ -direction then the condition  $\mathcal{D}_\ell = 0$  is automatically satisfied when  $\ell = 0$ . In this section, this argument is sketched for the case of a  $y$ -direction reflection symmetry. A similar argument works in any transverse direction.

The implication of a reflection symmetry for the solution set is that  $Z(x, -y, z, t)$  is a solution whenever  $Z(x, y, z, t)$  is a solution. This reflection symmetry will also arise in some form in the functions  $\mathcal{A}(\omega, k, m, \ell)$  and  $\mathcal{B}(\omega, k, m, \ell)$ . In fact, we will show that an implication of transverse  $y$ -reflection is the following property:

$$\left. \begin{aligned} \mathcal{A}(\omega, k, -m, \ell) &= \mathcal{A}(\omega, k, m, \ell), & \mathcal{B}(\omega, k, -m, \ell) &= \mathcal{B}(\omega, k, m, \ell), \\ \mathcal{C}(\omega, k, -m, \ell) &= -\mathcal{C}(\omega, k, m, \ell) & \text{and} & \mathcal{D}(\omega, k, -m, \ell) = \mathcal{D}(\omega, k, m, \ell). \end{aligned} \right\} \quad (7.1)$$

Hence,  $\mathcal{A}$ ,  $\mathcal{B}$  and  $\mathcal{D}$  are even functions of  $m$  and  $\mathcal{C}$  is an odd function of  $m$ .

A system in multisymplectic form (3.1), is *transverse reversible* in the  $y$ -direction if there exists a *reversor*  $\mathbf{R}$  acting on  $\mathbb{R}^n$  satisfying

$$\mathbf{R}\mathbf{M} = \mathbf{M}\mathbf{R}, \quad \mathbf{R}\mathbf{J} = \mathbf{J}\mathbf{R}, \quad \mathbf{R}\mathbf{K} = -\mathbf{K}\mathbf{R}, \quad \mathbf{R}\mathbf{P} = \mathbf{P}\mathbf{R} \quad \text{and} \quad S(\mathbf{R}\mathbf{Z}) = S(\mathbf{Z}). \quad (7.2)$$

An operator  $\mathbf{R}$  is a *reversor* if it is an involution and an isometry.

Act on (3.1) with  $\mathbf{R}$ ,

$$\mathbf{R}\mathbf{M}\mathbf{Z}_t + \mathbf{R}\mathbf{J}\mathbf{Z}_x + \mathbf{R}\mathbf{K}\mathbf{Z}_y + \mathbf{R}\mathbf{P}\mathbf{Z}_z = \mathbf{R}\nabla S(\mathbf{Z}),$$

and use (7.2),

$$\mathbf{M}(\mathbf{R}\mathbf{Z})_t + \mathbf{J}(\mathbf{R}\mathbf{Z})_x - \mathbf{K}(\mathbf{R}\mathbf{Z})_y + \mathbf{P}(\mathbf{R}\mathbf{Z})_z = \nabla S(\mathbf{R}\mathbf{Z}).$$

An immediate implication is that  $\mathbf{R}\mathbf{Z}(x, -y, z, t)$  is a solution of (3.1) whenever  $Z(x, y, z, t)$  is a solution.

We will verify the third of (7.1) as it is the most important with the verification of the others following a similar argument. Start with the definition

$$\mathcal{C}(\omega, k, m, \ell) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{2} \langle \mathbf{K} \hat{Z}_\theta, \hat{Z} \rangle d\theta, \quad (7.3)$$

with  $\hat{Z}(\theta, \omega, k, m, \ell)$  satisfying (3.2). Act on this equation with  $\mathbf{R}$  and use (7.2) to establish that

$$\mathbf{R} \hat{Z}(\theta, \omega, k, m, \ell) = \hat{Z}(\theta, \omega, k, -m, \ell). \quad (7.4)$$

Now, use this identity in (7.3),

$$\begin{aligned} \mathcal{C}(\omega, k, -m, \ell) &= \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{2} \langle \mathbf{K} \frac{d}{d\theta} \hat{Z}(\theta, \omega, k, -m, \ell), \hat{Z}(\theta, \omega, k, -m, \ell) \rangle d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{2} \langle \mathbf{K} \frac{d}{d\theta} \mathbf{R} \hat{Z}(\theta, \omega, k, m, \ell), \mathbf{R} \hat{Z}(\theta, \omega, k, m, \ell) \rangle d\theta \\ &= -\frac{1}{2\pi} \int_0^{2\pi} \frac{1}{2} \langle \mathbf{R} \mathbf{K} \frac{d}{d\theta} \hat{Z}(\theta, \omega, k, m, \ell), \mathbf{R} \hat{Z}(\theta, \omega, k, m, \ell) \rangle d\theta \\ &= -\frac{1}{2\pi} \int_0^{2\pi} \frac{1}{2} \langle \mathbf{K} \frac{d}{d\theta} \hat{Z}(\theta, \omega, k, m, \ell), \hat{Z}(\theta, \omega, k, m, \ell) \rangle d\theta \\ &= -\mathcal{C}(\omega, k, m, \ell), \end{aligned}$$

since  $\mathbf{R} \mathbf{K} = -\mathbf{K} \mathbf{R}$  and  $\mathbf{R}$  is an isometry. This completes the verification of the third identity in (7.1), with the others verified in a similar manner. The fact that  $\mathcal{C}$  is an odd function of  $m$  gives immediately

$$\mathcal{C}_k|_{m=0} = 0.$$

In other words, the emergence of KP has the same codimension as KdV in systems with a transverse symmetry, obtained by just using a basic state aligned with the  $x$ -direction. The example of NLS in  $3 + 1$  in the next section has a reflection symmetry in all transverse directions.

## 8. Example: $3 + 1$ nonlinear Schrödinger equation

Consider the defocusing NLS equation in  $3 + 1$  in standard form

$$i\psi_t + \psi_{xx} + \psi_{yy} + \psi_{zz} + \psi - |\psi|^2\psi = 0, \quad (8.1)$$

for the complex-valued function  $\psi(x, y, z, t)$ . This system is a basis for the discussion of quantum vortices [15]. Separate the equation into real and imaginary parts by letting  $\psi = a_1 + ia_2$ , giving

$$\text{and } \left. \begin{aligned} -\frac{\partial a_2}{\partial t} + \frac{\partial^2 a_1}{\partial x^2} + \frac{\partial^2 a_1}{\partial y^2} + \frac{\partial^2 a_1}{\partial z^2} + a_1 - (a_1^2 + a_2^2)a_1 &= 0 \\ \frac{\partial a_1}{\partial t} + \frac{\partial^2 a_2}{\partial x^2} + \frac{\partial^2 a_2}{\partial y^2} + \frac{\partial^2 a_2}{\partial z^2} + a_2 - (a_1^2 + a_2^2)a_2 &= 0. \end{aligned} \right\} \quad (8.2)$$

These two equations are the Euler-Lagrange equation associated with the Lagrangian

$$\begin{aligned} \mathcal{L}(\mathbf{a}, \mathbf{a}_x, \mathbf{a}_y, \mathbf{a}_z, \mathbf{a}_t) &= \frac{1}{2} \left( a_1 \frac{\partial a_2}{\partial t} - a_2 \frac{\partial a_1}{\partial t} + \left( \frac{\partial a_1}{\partial x} \right)^2 + \left( \frac{\partial a_1}{\partial y} \right)^2 \right. \\ &\quad \left. + \left( \frac{\partial a_1}{\partial z} \right)^2 + \left( \frac{\partial a_2}{\partial x} \right)^2 + \left( \frac{\partial a_2}{\partial y} \right)^2 + \left( \frac{\partial a_2}{\partial z} \right)^2 \right. \\ &\quad \left. - (a_1^2 + a_2^2) + \frac{1}{2} (a_1^2 + a_2^2)^2 \right). \end{aligned} \quad (8.3)$$

The multisymplectic form of the equations is obtained by taking sequential Legendre transforms

$$\mathbf{b} = \frac{\partial \mathcal{L}}{\partial \mathbf{a}_x} = \mathbf{a}_x, \quad \mathbf{c} = \frac{\partial \mathcal{L}}{\partial \mathbf{a}_y} = \mathbf{a}_y \quad \text{and} \quad \mathbf{d} = \frac{\partial \mathcal{L}}{\partial \mathbf{a}_z} = \mathbf{a}_z,$$

where  $\mathbf{a} = (a_1, a_2)^T$ . This then allows us to write the Lagrangian for this system in the form of (3.1) with

$$\left. \begin{aligned} Z &= \begin{pmatrix} \mathbf{a} \\ \mathbf{b} \\ \mathbf{c} \\ \mathbf{d} \end{pmatrix} \quad \mathbf{M} = \begin{pmatrix} \sigma & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{J} = \begin{pmatrix} 0 & -\mathbf{I} & 0 & 0 \\ \mathbf{I} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ \mathbf{K} &= \begin{pmatrix} 0 & 0 & -\mathbf{I} & 0 \\ 0 & 0 & 0 & 0 \\ \mathbf{I} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{P} = \begin{pmatrix} 0 & 0 & 0 & -\mathbf{I} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \mathbf{I} & 0 & 0 & 0 \end{pmatrix} \end{aligned} \right\} \quad (8.4)$$

and 
$$S(Z) = \frac{1}{2}(\mathbf{a} \cdot \mathbf{a} + \mathbf{b} \cdot \mathbf{b} + \mathbf{c} \cdot \mathbf{c} + \mathbf{d} \cdot \mathbf{d}) - \frac{1}{4}(\mathbf{a} \cdot \mathbf{a})^2, \quad \sigma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The associated conservation law of the form (1.7) can be deduced from (8.1) with

$$A = \frac{1}{2}|\psi|^2, \quad B = \Im(\psi^* \psi_x), \quad C = \Im(\psi^* \psi_y) \quad \text{and} \quad D = \Im(\psi^* \psi_z), \quad (8.5)$$

where  $\psi^*$  denotes the complex conjugate of  $\psi$  and  $\Im$  denotes the imaginary part of the expression. This conservation law can be verified by direct calculation using (8.1). The symbols  $(A, B, C, D)$  represent the components of the conservation law as functions of  $\psi$ . For the modulation theory, it is these components evaluated on the basic state that are important.

Consider a basic state of the form

$$\hat{\psi}(\theta, k, m, \ell, \omega) = \Psi_0 e^{i(kx + my + \ell z + \omega t)} = \Psi_0 e^{i\theta}.$$

The vector  $\hat{Z}(\theta, k, m, \ell, \omega)$  is obtained by defining  $\hat{Z} = (\mathbf{a}, k\mathbf{a}_\theta, m\mathbf{a}_\theta, \ell\mathbf{a}_\theta)$  with  $\mathbf{a} = (a_1, a_2)$  and  $a_1 + ia_2 = \Psi_0 e^{i\theta}$ . This state is an exact periodic travelling wave solution and substitution into the governing equation gives

$$\omega + k^2 + m^2 + \ell^2 - 1 + |\Psi_0|^2 = 0, \quad \Rightarrow \quad |\Psi_0|^2 = 1 - \omega - k^2 - m^2 - \ell^2. \quad (8.6)$$

To determine the derivatives for the necessary condition and the coefficients of the emergent KP equation, substitute into the components of the conservation law, giving

$$\left. \begin{aligned} \mathcal{A} &= \frac{1}{2}(1 - \omega - k^2 - m^2 - \ell^2), \quad \mathcal{B} = k(1 - \omega - k^2 - m^2 - \ell^2), \\ \mathcal{C} &= m(1 - \omega - k^2 - m^2 - \ell^2) \quad \text{and} \quad \mathcal{D} = \ell(1 - \omega - k^2 - m^2 - \ell^2). \end{aligned} \right\} \quad (8.7)$$

These expressions can also be obtained using the method to derive (3.3), however in this case it appears to be far simpler to extract the conservation law for the system directly and evaluate along the basic state. Note that  $\mathcal{A}, \mathcal{B}, \mathcal{D}$  are even functions of  $m$  and  $\mathcal{C}$  is an odd function of  $m$ . Similarly,  $\mathcal{A}, \mathcal{B}, \mathcal{C}$  are even functions of  $\ell$  and  $\mathcal{D}$  is an odd function of  $\ell$ . These symmetries are due to the transverse reflection symmetry of 3 + 1 NLS in the  $y$ - and  $z$ -directions, respectively. These symmetries follow from the theory in §7, or simply by noting that the  $y$ - and  $z$ -derivatives in 3 + 1 NLS (8.1) are even.

Seeking  $k$  extremality in  $\mathcal{B}, \mathcal{C}, \mathcal{D}$  gives the set of equations

$$1 - m^2 - \ell^2 - 3k^2 - \omega = 0, \quad mk = 0, \quad \ell k = 0. \quad (8.8)$$

To avoid the trivial solution,  $k = 0$ , the second two conditions give  $m = \ell = 0$ . The first condition then reduces to

$$k^2 = \frac{1}{3}(1 - \omega) \quad \text{with} \quad \omega < 1.$$

The derivatives of the conservation laws that appear in the emergent KP take the values

$$\mathcal{A}_k = -k, \quad \mathcal{B}_{kk} = -6k, \quad \mathcal{C}_m = 2k^2 \quad \text{and} \quad \mathcal{D}_\ell = 2k^2. \quad (8.9)$$

All that remains is to compute the  $x$ -direction dispersion coefficient  $\mathcal{K}$  in order to be able to construct the emergent KP equation. For this coefficient, the symplectic Jordan chain needs to be constructed.

Express the Jordan chain elements in the form

$$\xi_j = \begin{pmatrix} R_\theta \mathbf{a}_j \\ R_\theta \mathbf{b}_j \\ R_\theta \mathbf{c}_j \\ R_\theta \mathbf{d}_j \end{pmatrix}, \quad j = 1 \dots 4, \quad R_\theta = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix},$$

as well as the form for the state vector

$$\hat{Z} = \begin{pmatrix} R_\theta \hat{\mathbf{u}} \\ k\sigma R_\theta \hat{\mathbf{u}} \\ m\sigma R_\theta \hat{\mathbf{u}} \\ \ell\sigma R_\theta \hat{\mathbf{u}} \end{pmatrix},$$

so that  $|\hat{\mathbf{u}}| = |\Psi_0|$ . In the above, we have used that  $(d/d\theta)R_\theta = \sigma R_\theta$  and the latter expression is commutative. The sequence that produces the elements of the Jordan chain can be found to be

$$\begin{aligned} -2(\mathbf{a}_j \cdot \hat{\mathbf{u}})\hat{\mathbf{u}} &= -\mathbf{b}_{j-1} - k\sigma \mathbf{a}_{j-1}, \\ \mathbf{b}_j &= k\sigma \mathbf{a}_j + \mathbf{a}_{j-1}, \\ \mathbf{c}_j &= m\sigma \mathbf{a}_j, \\ \mathbf{d}_j &= \ell\sigma \mathbf{a}_j. \end{aligned}$$

Taking the initial vectors  $\mathbf{a}_0 = \mathbf{b}_0 = \mathbf{c}_0 = \mathbf{d}_0 = \mathbf{0}$ , then we find that

$$\left. \begin{aligned} \xi_1 &= \begin{pmatrix} \sigma R_\theta \hat{\mathbf{u}} \\ -kR_\theta \hat{\mathbf{u}} \\ -mR_\theta \hat{\mathbf{u}} \\ -\ell R_\theta \hat{\mathbf{u}} \end{pmatrix}, \quad \xi_2 = \frac{1}{2k} \begin{pmatrix} -R_\theta \hat{\mathbf{u}} \\ k\sigma R_\theta \hat{\mathbf{u}} \\ -m\sigma R_\theta \hat{\mathbf{u}} \\ -\ell\sigma R_\theta \hat{\mathbf{u}} \end{pmatrix}, \\ \xi_3 &= -\frac{1}{2k} \begin{pmatrix} \mathbf{0} \\ R_\theta \hat{\mathbf{u}} \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix} \quad \text{and} \quad \xi_4 = -\frac{1}{8k^3} \begin{pmatrix} R_\theta \hat{\mathbf{u}} \\ k\sigma R_\theta \hat{\mathbf{u}} \\ m\sigma R_\theta \hat{\mathbf{u}} \\ \ell\sigma R_\theta \hat{\mathbf{u}} \end{pmatrix}. \end{aligned} \right\} \quad (8.10)$$

This allows us to calculate  $\mathcal{K}$  as

$$\mathcal{K} = \langle \mathbf{J} \xi_1, \xi_4 \rangle = -\frac{1}{2}. \quad (8.11)$$

Therefore, the emergent KP equation for this problem is given by

$$(2kq_T + 6kq q_X + \frac{1}{2}q_{XXX})_X - 2k^2(q_{YY} + q_{ZZ}) = 0 \quad (8.12)$$

which is KP-I, with the transverse dispersion the same in the  $y$ - and  $z$ -directions. By scaling  $q, X, Y, Z, T$  appropriately, the emergent KP equation (8.12) can be put into canonical form

$$(q_T + q q_X + q_{XXX})_X - (q_{YY} + q_{ZZ}) = 0. \quad (8.13)$$

This equation has been studied by Senatorski & Infeld [4] and Infeld *et al.* [5] and localized solitary waves in three space dimensions have been shown to form from perturbed exact two-dimensional lump solitons.

## 9. Emergence of Kadomtsev–Petviashvili in $N > 3$ space dimensions

The emergence of KP follows from two key structural properties: the multisymplectic form of the Euler–Lagrange equation (3.1), and the conservation of wave action in geometric form (3.3). Of secondary importance is the scaling and the necessary conditions (1.9). All of these requirements generalize to arbitrary space dimension. Although applications in space dimension  $N > 3$  are not obvious, it is straightforward to sketch the argument leading to KP in  $N + 1$ .

Consider the following generalization of (3.1):

$$\mathbf{M}Z_t + \mathbf{J}Z_x + \sum_{n=1}^{N-1} \mathbf{K}_n Z_{x_n} = \nabla S(Z), \quad Z \in \mathbb{R}^n, \quad (9.1)$$

where the vector of spatial variables is  $\mathbf{x} = (x, x_1, \dots, x_{N-1})$  and  $\mathbf{M}, \mathbf{J}, \mathbf{K}_n$ ,  $n = 1, \dots, N - 1$ , are skew-symmetric matrices. A periodic travelling wave is of the form

$$Z(\mathbf{x}, t) = \hat{Z}(\mathbf{k} \cdot \mathbf{x} + \omega t + \theta_0) \equiv \hat{Z}(\theta),$$

for wavenumber vector  $\mathbf{k} = (k, k_1 \dots k_{N-1})$ , frequency  $\omega$  and phase shift  $\theta_0$ . Substitution of this ansatz into (9.1) results in the ODE

$$\left( \omega \mathbf{M} + k \mathbf{J} + \sum_{n=1}^{N-1} k_n \mathbf{K}_n \right) \hat{Z}_\theta = \nabla S(\hat{Z}). \quad (9.2)$$

The components of the conservation law for wave action have the natural generalization

$$\mathcal{A} = \frac{1}{4\pi} \int_0^{2\pi} \langle \mathbf{M}Z_\theta, Z \rangle d\theta, \quad \mathcal{B} = \frac{1}{4\pi} \int_0^{2\pi} \langle \mathbf{J}Z_\theta, Z \rangle d\theta \quad \text{and} \quad \mathcal{C}_i = \frac{1}{4\pi} \int_0^{2\pi} \langle \mathbf{K}_i Z_\theta, Z \rangle d\theta,$$

for  $i = 1, \dots, N - 1$ .

Generalizing the modulation ansatz (1.4), introducing slow space scales of order  $\varepsilon^2$  in all the transverse directions, substituting into the Euler–Lagrange equation and computing terms up to fifth order in  $\varepsilon$ , leads to the following necessary conditions:

$$\mathcal{B}_k = (\mathcal{C}_1)_k = \dots = (\mathcal{C}_{N-1})_k = 0,$$

and the following generalization of KP at fifth order:

$$(2\mathcal{A}_k q_T + \mathcal{B}_{kk} q q_X + \mathcal{K} q_{XXX})_X + \sum_{n=1}^{N-1} (\mathcal{C}_n)_k q_{X_n X_n} = 0. \quad (9.3)$$

**Data Accessibility.** This paper is comprised of theory only and contains no data.

**Authors' Contributions.** The authors contributed equally. The germ of the idea came from the second author. The first author undertook the analysis and example, both checked by the second. Both authors contributed to the writing.

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## References

1. Kadomtsev BB, Petviashvili VI. 1970 On the stability of solitary waves in weakly dispersing media. *Sov. Phys. Dokl* **15**, 539–541.
2. Biondini G, Pelinovsky D. 2008 Kadomtsev–Petviashvili equation. *Scholarpedia* **3**, 6539. (doi:10.4249/scholarpedia.6539)
3. Kuznetsov EA, Turitsyn K. 1982 Two- and three-dimensional solitons in weakly dispersive media. *Sov. Phys. JETP* **55**, 844–847.
4. Senatorski A, Infeld E. 1998 Breakup of two-dimensional into three-dimensional Kadomtsev–Petviashvili solitons. *Phys. Rev. E* **57**, 6050–6055. (doi:10.1103/PhysRevE.57.6050)

5. Infeld E, Rowlands G, Senatorski A. 1999 Instabilities and oscillations of one- and two-dimensional Kadomtsev–Petviashvili waves and solitons. *Proc. R. Soc. Lond. A* **455**, 4363–4381. (doi:10.1098/rspa.1999.0505)
6. Infeld E, Rowlands G. 1990 *Nonlinear waves, solitons and chaos*. Cambridge, UK: Cambridge University Press.
7. Ma W-X. 2011 Comment on the 3 + 1 dimensional Kadomtsev–Petviashvili equations. *Commun. Nonlinear Sci. Numer. Simulat.* **16**, 2663–2666. (doi:10.1016/j.cnsns.2010.10.003)
8. Whitham GB. 2011 *Linear and nonlinear waves*, vol. 42. New York, NY: John Wiley & Sons.
9. Doelman A, Sandstede B, Scheel A, Schneider G. 2009 *The dynamics of modulated wave trains*. Providence, RI: American Mathematical Society.
10. Bridges TJ. 2013 A universal form for the emergence of the Korteweg–de Vries equation. *Proc. R. Soc. A* **469**, 20120707. (doi:10.1098/rspa.2012.0707)
11. Bridges TJ. 2014 Emergence of dispersion in shallow water hydrodynamics via modulation of uniform flow. *J. Fluid Mech.* **761**, R1. (doi:10.1017/jfm.2014.653)
12. Bridges TJ. 2015 *Symmetry, modulation and nonlinear waves* (eds TJ Bridges, M Groves, DP Nicholls). Lectures on the Theory of Water Waves, pp. 1–20. Cambridge, UK: Cambridge University Press.
13. Wise FW. 2010 Generation of light bullets. *Physics* **3**, 107. (doi:10.1103/Physics.3.107)
14. Jones CA, Roberts PH. 1982 Motions in a bose condensate: IV. Axisymmetric solitary waves. *J. Phys. A: Math. Gen.* **15**, 2599–2619. (doi:10.1088/0305-4470/15/8/036)
15. Kerr RM. 2011 Vortex stretching as a mechanism for quantum kinetic energy decay. *Phys. Rev. Lett.* **106**, 224501. (doi:10.1103/PhysRevLett.106.224501)
16. Chiron D, Rousset F. 2010 The KdV/KP-I limit of the nonlinear Schrödinger equation. *SIAM J. Math. Anal.* **42**, 64–96. (doi:10.1137/080738994)
17. Pelinovsky DE, Stepanyants YA, Kivshar YS. 1995 Self-focusing of plane dark solitons in nonlinear defocusing media. *Phys. Rev. E* **51**, 5016. (doi:10.1103/PhysRevE.51.5016)
18. Chirilus-Bruckner M, Düll W-P, Schneider G. 2014 Validity of the KdV equation for the modulation of periodic traveling waves in the NLS equation. *J. Math. Anal. Appl.* **414**, 166–175. (doi:10.1016/j.jmaa.2013.12.061)
19. Bridges TJ. 1997 Multisymplectic structure and wave propagation. *Math. Proc. Camb. Phil. Soc.* **121**, 147–190. (doi:10.1017/S0305004196001429)
20. Bridges TJ, Hydon PE, Lawson JK. 2010 Multisymplectic structure and the variational bicomplex. *Math. Proc. Camb. Phil. Soc.* **148**, 159–178. (doi:10.1017/S0305004109990259)
21. Bridges TJ. 1997 A geometrical formulation of the conservation of wave action and its implications for signature and the classification of instabilities. *Proc. R. Soc. Lond. A* **453**, 1365–1395. (doi:10.1098/rspa.1997.0075)