# Non−flocking and flocking for the Cucker−Smale model with distributed time delays

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**Abstract**

In this paper, we study a flocking behavior that may or not appear for Cucker−Smale model with distributed time delays. For the short range communicated Cucker−Smale model, the flocking condition has strong restrictions on initial data. For this case, we mainly consider the non−flocking behavior. By establishing and appropriately estimating an inequality of the position variance such that the second order space moment is unbounded, we drive a sufficient condition for the non−existence of the asymptotic flocking when the time delays satisfy a suitable smallness assumption. Furthermore, we also provide a sufficient condition of asymptotic flocking. Finally, we present numerical simulations to validate the theoretical results.

*Keywords:* Cucker−Smale model, distributed time delays, non−flocking, flocking.

## 1. Introduction

The Cucker−Smale(C−S) model is one of the most celebrated multi-agent models, which can describe the flocking behavior of moving animals in nature, such as flying of birds, herding of sheep and schooling of fishes [1–3]. This model was first proposed by Cucker and Smale in 2007 [1], then it was quickly extended in many directions, including the model with time delays in [4–10], the model with collisions avoidance in [11–16], the model with nonlinear systems in [17–19], the model with the problem of leader in [20, 21], and so forth.

In a real multi-agent system, it is natural to introduce a delay as a reaction time or a time to receive environmental information, which is denoted as the reaction delay or the transmission delay respectively. The C−S model and other extended models with the transmission delay were investigated in [22, 23]. However, in typical applications of C−S model in biology or engineering, the transmission delay is much smaller than the reaction delay due to the high speed communication of agents. Thus, we pay attention on the C−S model with reaction delays. When this delay is a positive constant, [24] proved that there is an asymptotic alignment of velocities for the delayed C−S model with or without noise, under the a-priori assumption that the Fiedler number (smallest positive eigenvalue) of the communication matrix is uniformly bounded away from zero. By using similar assumptions of the Fiedler number of Laplacian matrix, some sufficient and/or necessary conditions of controllability for the non-delay systems were derived in [25, 26]. Then, Haskovec and Markou derived sufficient conditions of flocking behavior in [27] for all classical communication weights. In a more realistic situation, the time delay is distributed over an interval rather than

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concentrated at a single time instant. Thus, the C−S model with distributed reaction delays was further considered in [27, 28]. However, to get the flocking behavior, the decreasing of the time−varying delay or a lower bound of the derivative of the communication weight is assumed in these papers, which is not natural. In this paper, these assumptions are successfully removed, and the flocking behavior is also established. For other flocking results on delayed C−S models, we refer to [22, 23, 28–30] and the references therein.

Note that the above results only focus on the flocking behavior. But, the non−flocking behavior can be easily observed in numerical simulations. It is also common in many practical situations, such as the opinion disagreement in society, fish flock breaking, flight multi-formation and so on. Moreover, for the short range communicated C−S model ($\exists r_0 > 0$ such that $\int_0^{r_0} \phi(r)dr = \infty$; where $\phi$ is a communication weight function), it is unreasonable to directly investigate sufficient conditions of flocking, since the conditions have strong restrictions on initial data. In other words, it is more appropriate to study the non-flocking behavior. However, the theoretical results are far from perfect particularly about the delay model. Recently, it is shown in [31] that the C−S model has the non-flocking behavior when the initial data satisfies some conditions. When there exist distributed delays, we are also interested to find a sufficient condition of the non-flocking behavior, which should be simple enough. We refer to [32–35] for other non-flocking or multi-flocking results.

The remainder of this paper is organized as follows. Section 2 is to address system description and some preliminaries. Section 3 is devoted to giving a simple sufficient condition to the non-flocking behavior. In Section 4, we also establish the flocking behavior for the delayed model under some initial conditions. In Section 5, some simulation examples are shown to validate our theoretical results. For simplicity, some proofs are given in Appendix.

## 2. Preliminaries

Let $N$ be the number of agents, and $(x_i(t), v_i(t)) \in \mathbb{R}^{2d}$ denote the position and velocity of $i$th agent at the time $t$. Let the communication weight $\phi$ be positive and Lipschitz. Then, the delayed model is described by

$$\begin{cases} \dot{x}_i = v_i, \\ \dot{v}_i = \dfrac{\lambda}{Nh(t)} \displaystyle\sum_{k=1}^{N} \int_{t-\tau(t)}^{t} \alpha(t-s)\phi(|x_k(s) - x_i(s)|)(v_k(s) - v_i(s))ds, \end{cases} \tag{1}$$

where the parameter $\lambda > 0$ measures the alignment strength, $\tau(t) \geq 0$ satisfies that $0 < \tau_0 \leq \tau(t) \leq \bar{\tau}$ for all $t \geq 0$. Moreover, $\alpha : [0, \bar{\tau}] \to [0, \infty)$ is a weight function satisfying

$$h(t) := \int_0^{\tau(t)} \alpha(s)ds, \qquad \int_0^{\tau_0} \alpha(s)ds > 0.$$

For the above model, we consider prescribed initial position and velocity trajectories

$$(x_i(t), v_i(t)) = (x_i^0(t), v_i^0(t)), \quad \forall\, t \in [-\bar{\tau}, 0], \tag{2}$$

where $x_i^0, v_i^0 \in C([-\bar{\tau}, 0]; \mathbb{R}^d)$.

For simplicity, we define the position and velocity fluctuations as

$$V(t) := \left( \sum_{i=1}^{N} \sum_{k=1}^{N} |v_k(t) - v_i(t)|^2 \right)^{\frac{1}{2}}, \quad X(t) := \left( \sum_{i=1}^{N} \sum_{k=1}^{N} |x_k(t) - x_i(t)|^2 \right)^{\frac{1}{2}}. \tag{3}$$

By defining the average position and velocity

$$(x_c(t), v_c(t)) := \left(\frac{1}{N}\sum_{i=1}^{N} x_i(t), \frac{1}{N}\sum_{i=1}^{N} v_i(t)\right),$$

we know from (1) that $v_c(t) \equiv v_c(0)$ for any $t \geq 0$, and then $x_c(t) = x_c(0) + v_c(0)t$. Thus, we have that for any $t \geq 0$,

$$V(t) = \left(2N\sum_{i=1}^{N} |v_i - v_c|^2\right)^{\frac{1}{2}}, \qquad X(t) = \left(2N\sum_{i=1}^{N} |x_i - x_c|^2\right)^{\frac{1}{2}}, \tag{4}$$

but they don't hold for $t \in [-\bar{\tau}, 0)$ generally.

Firstly, a local classical solution $\{(x_i, v_i)\}_{i=1}^N$ of model (1), (2) can be obtained, since the right hand side of (1) is continuous as a function of $(x_i, v_i)$. Actually, its solution is global because the boundedness of $V(t)$ and $X(t)$ will be proved in Section 4.

Secondly, we give the definition of asymptotic flocking of model (1), (2).

**Definition 1.** *[1] Model (1), (2) exhibits asymptotic flocking if and only if*

$$\lim_{t\to\infty} V(t) = 0 \qquad \text{and} \qquad \sup_{t\geq 0} X(t) < \infty.$$

In particular, non-flocking (see [36] for details) can be classified in more details as semi-nonflocking, where only $\lim_{t\to\infty} V(t) = 0$ holds, and full non-flocking, where neither $\lim_{t\to\infty} V(t) = 0$ nor $\sup_{t\geq 0} X(t) < \infty$ holds. The non−existence of the asymptotic flocking is proved through the unboundedness of $X(t)$ in next section.

## 3. The non-flocking behavior

For simplicity, we define $\phi_{ki} = \phi(|x_k(s) - x_i(s)|)$ and $\Phi_{ki}(t) := \int_0^{|x_k(t)-x_i(t)|} r\phi(r)dr$, $\forall\ 1 \leq k, i \leq N$. We are now devoted to establishing a simple sufficient condition to the non-flocking behavior of model (1), (2).

**Theorem 1.** *Let $\tau(t)$ be a continuously differentiable function such that $\tau' < 1$, $\tau' \in L^1(\mathbb{R}^+)$ and $\tau_0 \leq \tau(t) \leq \bar{\tau}$. Assume that $\alpha$ is bounded, and $0 \leq \phi \in C_b(\mathbb{R}^+)$ and $r\phi(r) \in L^1(\mathbb{R}^+)$. Let $k_0 = \frac{\max_{s\in[-\bar{\tau},0]} V(s)}{V(0)}$, when $\bar{\tau}$ is sufficiently small such that $4k_0^2\lambda\|\phi\|_\infty\bar{\tau} \leq 1$, the classical solution $\{(x_i, v_i)\}_{i=1}^N$ of model (1), (2) exhibits non-flocking if*

$$\sum_{i=1}^{N}(x_i^0 - x_c)\cdot(v_i^0 - v_c) > \|r\phi(r)\|_{L^1}\left(\frac{\|\alpha\|_\infty}{\int_0^{\tau_0}\alpha(s)ds}\|\tau'\|_{L^1} + 1\right). \tag{5}$$

**Remark 1.** *Note that $\phi$ and $\tau$ do not need to be decreasing in the above theorem. The typical short range communication weights $\phi(r) \leq (1 + r^2)^{-\frac{\beta}{2}}$ with $\beta > 2$ satisfy the assumption that $r\phi(r) \in L^1$. Any smooth function with compact support also satisfies this assumption.*

### *3.1. An inequality of the position variance*

Based on Definition 1, we intend to show the unboundedness of $X(t)$, which yields the non-flocking behavior. Inspired by the key equality of the position variance established in [31, 37], we obtain the following proposition.

**Proposition 1.** *Assume that $\tau(t)$ be a continuously differentiable function such that $\tau_0 \le \tau(t) \le \bar{\tau}$, and let $\{(x_i, v_i)\}_{i=1}^N$ be a smooth solution to model (1), (2). Then, we have that for any $t \ge \bar{\tau}$*

$$
\begin{aligned}
&\frac{d}{dt}\left(\frac{1}{2}\sum_{i=1}^N |x_i(t) - x_c(t)|^2\right) \\
\ge\ & \sum_{i=1}^N (x_i(\bar{\tau}) - x_c)\cdot(v_i(\bar{\tau}) - v_c) - \frac{\lambda}{2N}\sum_{i=1}^N\sum_{k=1}^N \int_{\bar{\tau}}^t \frac{1}{h(s)}\int_{s-\tau(s)}^s \alpha(s-u)\Phi'_{ki}(u)du ds \\
&+\frac{1}{2N}\int_{\bar{\tau}}^t \left(V^2(s) - \lambda\|\phi\|_\infty \bar{\tau} \sup_{u\in[s-\bar{\tau},s]} V^2(u)\right)ds. \qquad (6)
\end{aligned}
$$

**Proof.** We first compute $\frac{d^2}{dt^2}\left(\frac{1}{2}\sum_{i=1}^N |x_i - x_c|^2\right)$ and then integrate it over $(\bar{\tau}, t)$. Following from (1) we obtain that for any $t \ge \bar{\tau}$,

$$
\begin{aligned}
&\frac{d^2}{dt^2}\left(\frac{1}{2}\sum_{i=1}^N |x_i - x_c|^2\right) = \frac{d}{dt}\left(\sum_{i=1}^N (x_i - x_c)\cdot(v_i - v_c)\right) \\
=\ & \sum_{i=1}^N |v_i - v_c|^2 - \frac{\lambda}{2N}\sum_{i=1}^N\sum_{k=1}^N \frac{1}{h(t)}\int_{t-\tau(t)}^t \alpha(t-s)\phi_{ki}\cdot(v_k(s) - v_i(s))\cdot(x_k(t) - x_i(t))ds, \qquad (7)
\end{aligned}
$$

To estimate the last term of the right hand side of (7), we can rewrite $x_k(t) - x_i(t)$ as

$$
x_k(t) - x_i(t) = x_k(s) - x_i(s) + x_k(t) - x_k(s) - (x_i(t) - x_i(s)) = x_k(s) - x_i(s) + \int_s^t (v_k - v_i)(\theta)d\theta.
$$

Note that the definition of $\Phi_{ki}(t)$ gives that $(x_k(t) - x_i(t))\cdot(v_k(t) - v_i(t))\phi(|x_k(t) - x_i(t)|) = \frac{d}{dt}\Phi_{ki}(t)$. Then, we can use the above equalities to obtain that

$$
\begin{aligned}
&\sum_{i=1}^N\sum_{k=1}^N \phi_{ki}\cdot(v_k(s) - v_i(s))\cdot(x_k(t) - x_i(t)) \\
=\ & \sum_{i=1}^N\sum_{k=1}^N \phi_{ki}\cdot(v_k(s) - v_i(s))\cdot(x_k(s) - x_i(s)) + \sum_{i=1}^N\sum_{k=1}^N \phi_{ki}\cdot(v_k(s) - v_i(s))\cdot\int_s^t (v_k(\theta) - v_i(\theta))d\theta \\
\le\ & \sum_{i=1}^N\sum_{k=1}^N \frac{d}{ds}\Phi_{ki}(s) + \|\phi\|_\infty \bar{\tau} \sup_{u\in[t-\bar{\tau},t]} V^2(u), \qquad (8)
\end{aligned}
$$

since the Hölder inequality yields that

$$
\begin{aligned}
&\sum_{i,k=1}^N (v_k(s) - v_i(s))\cdot\int_s^t (v_k(\theta) - v_i(\theta))d\theta \\
\le\ & \int_s^t \left(\sum_{i,k=1}^N (v_k(s) - v_i(s))^2\right)^{\frac{1}{2}}\left(\sum_{i,k=1}^N (v_k(\theta) - v_i(\theta))^2\right)^{\frac{1}{2}} d\theta = \int_s^t V(s)V(\theta)d\theta \le \bar{\tau} \sup_{u\in[t-\bar{\tau},t]} V^2(u)
\end{aligned}
$$

for any $s \in [t - \tau(t), t]$. As a consequence, combining (7) with (8) we get that

$$
\begin{aligned}
&\frac{d^2}{dt^2}\left(\frac{1}{2}\sum_{i=1}^N |x_i - x_c|^2\right) \\
\ge\ & \frac{1}{2N}V^2(t) - \frac{\lambda}{2N}\sum_{i=1}^N\sum_{k=1}^N \frac{1}{h(t)}\int_{t-\tau(t)}^t \alpha(t-s)\Phi'_{ki}(s)ds - \frac{\lambda\|\phi\|_\infty \bar{\tau}}{2N} \sup_{u\in[t-\bar{\tau},t]} V^2(u), \qquad (9)
\end{aligned}
$$

since $\int_0^{\tau(t)} \alpha(s)ds = h(t)$. Finally, by integrating both sides of (9) over $[\bar{\tau}, t]$ we complete the proof. □

Taking into account the right hand side of (6), we firstly estimate the third term. In the following lemma, we show that $V(t)$ does not change dramatically when $t$ grows, so

$$V^2(s) - \lambda\|\phi\|_\infty \bar{\tau} \sup_{u \in [s-\bar{\tau}, s]} V^2(u)$$

can be non-negative when $\bar{\tau}$ is small enough.

**Lemma 1.** *Let $0 \le \phi \in C_b(\mathbb{R}^+)$, and let $V(0) \neq 0$. Assume that $\bar{\tau} > 0$ satisfies $e^{k\bar{\tau}} \le 2$, then*

$$V(s) \le k_0 e^{k(t-s)} V(t), \quad \forall\, t > 0, s \in [-\bar{\tau}, 0), \tag{10}$$

*and*

$$V(s) \le e^{k|t-s|} V(t), \quad \forall\, t, s \ge 0, \tag{11}$$

*where*

$$k_0 := \frac{\max_{s \in [-\bar{\tau}, 0]} V(s)}{V(0)}, \qquad k = 2k_0 \lambda \|\phi\|_\infty. \tag{12}$$

*As a result, for any $t \in (0, \bar{\tau}]$*

$$\frac{1}{2} V(0) \le V(t) \le 2V(0), \; X(t) \le X(0) + 2\bar{\tau} V(0). \tag{13}$$

By establishing the following delay inequality of $V$,

$$\frac{d}{dt} V(t) \ge -\lambda \|\phi\|_\infty \sup_{s \in [t-\tau(t), t]} V(s),$$

we can eventually prove the above lemma. The precise proof is given in Appendix.

According to the first term of the right hand side of (6), we need to estimate $\sum_{i=1}^N (x_i(\bar{\tau}) - x_c) \cdot (v_i(\bar{\tau}) - v_c)$. Since $\bar{\tau}$ is sufficiently small, the following lemma shows that $\sum_{i=1}^N (x_i(\bar{\tau}) - x_c) \cdot (v_i(\bar{\tau}) - v_c)$ is close to $\sum_{i=1}^N (x_i^0 - x_c) \cdot (v_i^0 - v_c)$. The precise proof is given in Appendix.

**Lemma 2.** *Let $0 \le \phi \in C_b(\mathbb{R}^+)$, and let $V(0) \neq 0$. Assume that $\bar{\tau} > 0$ satisfies $e^{k\bar{\tau}} \le 2$, then the classical solution $\{(x_i, v_i)\}_{i=1}^N$ of model (1), (2) satisfies that*

$$\begin{aligned} &\sum_{i=1}^N (x_i(\bar{\tau}) - x_c) \cdot (v_i(\bar{\tau}) - v_c) \\ \ge\; &\sum_{i=1}^N (x_i^0 - x_c) \cdot (v_i^0 - v_c) + \frac{\bar{\tau}}{2N} \left( \frac{1}{4} V^2(0) - \frac{\lambda \|\phi\|_\infty (X(0) + 2\bar{\tau} V(0)) V(0)}{2k} \right). \end{aligned}$$

### *3.2. The key estimate*

Obviously, it is key to compute $\int_{\bar{\tau}}^t \frac{1}{h(s)} \int_{s-\tau(s)}^s \alpha(s-u) \Phi'_{ki}(u) du ds$ in the right hand side of (6). If we roughly replace $\Phi'_{ki}(u)$ by $\Phi'_{ki}(s)$ in this term, then

$$\int_{\bar{\tau}}^t \frac{1}{h(s)} \int_{s-\tau(s)}^s \alpha(s-u) \Phi'_{ki}(s) du ds = \int_{\bar{\tau}}^t \Phi'_{ki}(s) ds = \Phi_{ki}(t) - \Phi_{ki}(\bar{\tau}),$$

since $h(s) = \int_0^{\tau(s)} \alpha(u) du$. By the definition of $\Phi_{ki}$, we know that

$$0 \le \Phi_{ki}(t) \le \|r\phi(r)\|_{L^1}, \tag{14}$$

so there is a good estimate

$$\int_{\bar{\tau}}^{t}\frac{1}{h(s)}\int_{s-\tau(s)}^{s}\alpha(s-u)\Phi'_{ki}(s)duds\leq\|r\phi(r)\|_{L^1}.$$

But, we can not follow this method directly. Actually, even we can establish that $\Phi'_{ki}(u)\leq\Phi'_{ki}(s)+C(s-u)$, then

$$\begin{aligned}&\int_{\bar{\tau}}^{t}\frac{1}{h(s)}\int_{s-\tau(s)}^{s}\alpha(s-u)\Phi'_{ki}(u)duds\\ \leq\ &\int_{\bar{\tau}}^{t}\frac{1}{h(s)}\int_{s-\tau(s)}^{s}\alpha(s-u)\Big(\Phi'_{ki}(s)+C(s-u)\Big)duds\leq\|r\phi(r)\|_{L^1}+C\int_{\bar{\tau}}^{t}\tau(s)ds,\end{aligned}$$

which is not bounded. In the following lemma, we give a reasonable estimate of $\int_{\bar{\tau}}^{t}\frac{1}{h(s)}\int_{s-\tau(s)}^{s}\alpha(s-u)\Phi'_{ki}(u)duds$. For clarity, the whole proof is divided into four steps.

**Lemma 3.** *Let $\{(x_i,v_i)\}_{i=1}^{N}$ be a smooth solution to model (1), (2). Let $\tau(t)$ be a continuously differentiable function such that $\tau'<1$, $\tau'\in L^1(\mathbb{R}^+)$ and $\tau_0\leq\tau(t)\leq\bar{\tau}$. Assume that $\alpha$ is bounded, and $0\leq\phi\in C_b(\mathbb{R}^+)$ and $r\phi(r)\in L^1(\mathbb{R}^+)$. Then,*

$$\int_{\bar{\tau}}^{t}\frac{1}{h(s)}\int_{s-\tau(s)}^{s}\alpha(s-u)\Phi'_{ki}(u)duds\leq\|r\phi(r)\|_{L^1}\left(\frac{\|\alpha\|_\infty}{\int_0^{\tau_0}\alpha(s)ds}\|\tau'\|_{L^1}+1\right),\qquad\forall t\geq2\bar{\tau}.$$

**Proof.** *Step 1: Changing the order of integral.*

By integrating over $s$ first we get that

$$\int_{\bar{\tau}}^{t}\frac{1}{h(s)}\int_{s-\tau(s)}^{s}\alpha(s-u)\Phi'_{ki}(u)duds=\int_{\bar{\tau}-\tau(\bar{\tau})}^{t}\left(\int_{\max\{u,\bar{\tau}\}}^{\min\{g(u),t\}}\frac{\alpha(s-u)}{h(s)}ds\right)\Phi'_{ki}(u)du,\tag{15}$$

where $g$ is a bijective function from $(0,\infty)$ to $(\tau_0,\infty)$ such that

$$g(u)-\tau(g(u))=u\tag{16}$$

since $\tau'<1$. Then $g$ is strictly increasing and

$$u+\tau_0\leq g(u)\leq u+\bar{\tau}.\tag{17}$$

Furthermore, from (16) we have know that $g$ and $\tau$ are commutative, then

$$1-g'(u)=-\tau'(g(u))g'(u)=-g'(\tau(u))\tau'(u),\tag{18}$$

*Step 2: Integrating by parts.*

For simplicity, we denote $f(u)=\int_{\max\{u,\bar{\tau}\}}^{\min\{g(u),t\}}\frac{\alpha(s-u)}{h(s)}ds$. By (16), we have that $t+\tau_0\leq g(t)\leq t+\bar{\tau}$, so for any $t\geq\bar{\tau}$

$$f(t)=\int_{\max\{t,\bar{\tau}\}}^{\min\{g(t),t\}}\frac{\alpha(s-t)}{h(s)}ds=0.$$

Similarly, following from (16) we get that $g(\bar{\tau}-\tau(\bar{\tau}))=\bar{\tau}$, and then for any $t\geq\bar{\tau}$

$$f(\bar{\tau}-\tau(\bar{\tau}))=\int_{\max\{\bar{\tau}-\tau(\bar{\tau}),\bar{\tau}\}}^{\min\{g(\bar{\tau}-\tau(\bar{\tau})),t\}}\frac{\alpha(s-\bar{\tau}+\tau(\bar{\tau}))}{h(s)}ds=0.$$

Combing the above two equalities with (15), we have that

$$\begin{aligned}
&\int_{\bar\tau}^{t}\frac{1}{h(s)}\int_{s-\tau(s)}^{s}\alpha(s-u)\Phi_{ki}'(u)duds \\
=\ & f(t)\Phi_{ki}(t)-f(\bar\tau-\tau(\bar\tau))\Phi_{ki}(\bar\tau-\tau(\bar\tau))-\int_{\bar\tau-\tau(\bar\tau)}^{t}\Phi_{ki}(u)f'(u)du \\
=\ & -\int_{\bar\tau-\tau(\bar\tau)}^{t}\Phi_{ki}(u)f'(u)du. \qquad (19)
\end{aligned}$$

Note that

$$\frac{d}{du}f(u) = -\int_{\max\{u,\bar\tau\}}^{\min\{g(u),t\}}\frac{\alpha'(s-u)}{h(s)}ds-\frac{\alpha(0)}{h(u)}1_{[\bar\tau,\infty]}(u)+\frac{g'(u)\alpha(g(u)-u)}{h(g(u))}1_{g(u)<t}(u), \quad \forall t\ge 2\bar\tau, \qquad (20)$$

where $1_\Omega(u)$ is an indicative function such that $1_\Omega(u)=1$ when $u\in\Omega$, otherwise $1_\Omega(u)=0$.

*Step 3: Dividing the integral region.*

To compute $f'(u)$, the integral region $[\bar\tau-\tau(\bar\tau),t]$ is divided into $[\bar\tau-\tau(\bar\tau),\bar\tau]$, $[\bar\tau,g^{-1}(t)]$ and $[g^{-1}(t),t]$. Firstly, for $u\in[\bar\tau-\tau(\bar\tau),\bar\tau]$ we have from (20) that

$$\begin{aligned}
-\frac{d}{du}f(u) &= \int_{\bar\tau}^{g(u)}\frac{\alpha'(s-u)}{h(s)}ds-\frac{g'(u)\alpha(g(u)-u)}{h(g(u))} \\
&= \frac{\alpha(g(u)-u)}{h(g(u))}-\frac{\alpha(\bar\tau-u)}{h(\bar\tau)}-\frac{g'(u)\alpha(g(u)-u)}{h(g(u))}+\int_{\bar\tau}^{g(u)}\frac{\alpha(s-u)\tau'(s)\alpha(\tau(s))}{h^2(s)}ds \\
&\le (1-g'(u))\frac{\alpha(g(u)-u)}{h(g(u))}+\int_{\bar\tau}^{g(u)}\frac{\alpha(s-u)\tau'(s)\alpha(\tau(s))}{h^2(s)}ds, \qquad (21)
\end{aligned}$$

since $g(u)$ is increasing and

$$h'(s)=\frac{d}{dt}\left(\int_0^{\tau(t)}\alpha(s)ds\right)=\tau'(s)\alpha(\tau(s)).$$

Similarly, we have from (20) that for $u\in[\bar\tau,g^{-1}(t)]$,

$$\begin{aligned}
-\frac{d}{du}f(u) &= \int_{u}^{g(u)}\frac{\alpha'(s-u)}{h(s)}ds+\frac{\alpha(0)}{g(u)}-\frac{g'(u)\alpha(g(u)-u)}{h(g(u))} \\
&\le (1-g'(u))\frac{\alpha(g(u)-u)}{h(g(u))}+\int_{u}^{g(u)}\frac{\alpha(s-u)\tau'(s)\alpha(\tau(s))}{h^2(s)}ds,
\end{aligned}$$

and for $u\in[g^{-1}(t),t]$

$$-\frac{d}{du}f(u) = \int_{u}^{t}\frac{\alpha'(s-u)}{h(s)}ds+\frac{\alpha(0)}{g(u)}\le\frac{\alpha(t-u)}{h(t)}+\int_{u}^{t}\frac{\alpha(s-u)\tau'(s)\alpha(\tau(s))}{h^2(s)}ds.$$

Combining the above two inequalities with (21), we can conclude that

$$\begin{aligned}
&-\int_{\bar\tau-\tau(\bar\tau)}^{t}\Phi_{ki}(u)f'(u)du \\
\le\ & \int_{\bar\tau-\tau(\bar\tau)}^{t}\Phi_{ki}(u)\int_{\min\{u,\bar\tau\}}^{\max\{t,g(u)\}}\frac{\alpha(s-u)\tau'(s)\alpha(\tau(s))}{h^2(s)}dsdu+\int_{\bar\tau-\tau(\bar\tau)}^{g^{-1}(t)}\Phi_{ki}(u)(1-g'(u))\frac{\alpha(g(u)-u)}{h(g(u))}du \\
&+\int_{g^{-1}(t)}^{t}\Phi_{ki}(u)\frac{\alpha(t-u)}{h(t)}du. \qquad (22)
\end{aligned}$$

*Step 4: Final estimates.*

Thus, for the third term of the right hand side of (22) we have that

$$\int_{g^{-1}(t)}^{t} \Phi_{ki}(u)\frac{\alpha(t-u)}{h(t)}du \quad\leq\quad \|r\phi(r)\|_{L^1}\frac{\int_0^{t-g^{-1}(t)}\alpha(s)ds}{h(t)} = \|r\phi(r)\|_{L^1}, \tag{23}$$

since (16) gives that $t - g^{-1}(t) = \tau(t)$. For the second term of the right hand side of (22), we can use (14) and (18) to get that

$$\begin{aligned}&\int_{\bar\tau-\tau(\bar\tau)}^{g^{-1}(t)} \Phi_{ki}(u)(1-g'(u))\frac{\alpha(g(u)-u)}{h(g(u))}du = \int_{\bar\tau-\tau(\bar\tau)}^{g^{-1}(t)} -\tau'(g(u))g'(u)\frac{\alpha(g(u)-u)}{h(g(u))}\Phi_{ki}(u)du\\ \leq\ & \|r\phi(r)\|_{L^1}\frac{\|\alpha\|_\infty}{\inf h}\int_{\bar\tau-\tau(\bar\tau)}^{g^{-1}(t)}(\tau'(g(u)))^- g'(u)du \leq \|r\phi(r)\|_{L^1}\frac{\|\alpha\|_\infty}{\int_0^{\tau_0}\alpha(s)ds}\|(\tau')^-\|_{L^1},\end{aligned} \tag{24}$$

where $\rho^+ = \max\{\rho, 0\}$ and $\rho^- = \max\{-\rho, 0\}$. Now, we compute the first term of the right hand side of (22). By integrating over $u$ first, we get that

$$\begin{aligned}&\int_{\bar\tau-\tau(\bar\tau)}^{t}\Phi_{ki}(u)\int_{\min\{u,\bar\tau\}}^{\max\{t,g(u)\}}\frac{\alpha(s-u)\tau'(s)\alpha(\tau(s))}{h^2(s)}dsdu\\ \leq\ & \int_{\bar\tau}^{t}\left(\int_{s-\tau(s)}^{s}\Phi_{ki}(u)\alpha(s-u)du\right)\frac{(\tau'(s))^+\alpha(\tau(s))}{h^2(s)}duds\\ \leq\ & \|r\phi(r)\|_{L^1}\int_{\bar\tau}^{t}\frac{(\tau'(s))^+\alpha(\tau(s))}{h(s)}ds \leq \|r\phi(r)\|_{L^1}\frac{\|\alpha\|_\infty}{\int_0^{\tau_0}\alpha(s)ds}\|(\tau')^+\|_{L^1}.\end{aligned} \tag{25}$$

Combining (22) with (23)–(25), we obtain that

$$-\int_{\bar\tau-\tau(\bar\tau)}^{t}\Phi_{ki}(u)f'(u)du \leq \|r\phi(r)\|_{L^1}\left(\frac{\|\alpha\|_\infty}{\int_0^{\tau_0}\alpha(s)ds}\|\tau'\|_{L^1}+1\right),$$

since $|\tau'| = (\tau')^+ + (\tau')^-$. Thus, following from the above inequality and (19), we complete the whole proof. □

### 3.3. *The proof of Theorem 1*

Combining Proposition 1 with the above three lemmas, we can prove Theorem 1.

**Proof of Theorem 1.** Note that $4k_0^2\lambda\|\phi\|_\infty\bar\tau \leq 1 \Rightarrow e^{k\bar\tau} \leq 2$ since $k = 2k_0\lambda\|\phi\|_\infty\bar\tau$ and $k_0 \geq 1$. So for any $s \geq 0$, we can use (10) to get that

$$\lambda\|\phi\|_\infty\bar\tau \sup_{u\in[s-\bar\tau,s]} V^2(u) \leq \lambda\|\phi\|_\infty\bar\tau k_0^2 e^{2k\bar\tau}V^2(s) \leq 4k_0^2\lambda\|\phi\|_\infty\bar\tau V^2(s),$$

and then by (11) we have that

$$\begin{aligned}&\frac{1}{2N}\int_{\bar\tau}^{t}\left(V^2(s) - \lambda\|\phi\|_\infty\bar\tau\sup_{u\in[s-\bar\tau,s]}V^2(u)\right)ds\\ \geq\ & \frac{1-4k_0^2\lambda\|\phi\|_\infty\bar\tau}{2N}\int_{\bar\tau}^{t}V^2(s)ds \geq \frac{1-4k_0^2\lambda\|\phi\|_\infty\bar\tau}{2N}\cdot\frac{V^2(0)\left(e^{-2k\bar\tau}-e^{-2kt}\right)}{2k}\\ \geq\ & \frac{1-4k_0^2\lambda\|\phi\|_\infty\bar\tau}{4Nk}V^2(0)\left(\frac{1}{4}-e^{-2kt}\right).\end{aligned}$$

Note that

$$\frac{1-4k_0^2\lambda\|\phi\|_\infty\bar{\tau}}{16Nk}V^2(0)+\frac{\bar{\tau}}{2N}\left(\frac{1}{4}V^2(0)-\frac{\lambda\|\phi\|_\infty(X(0)+2\bar{\tau}V(0))V(0)}{2k}\right)>0$$

for sufficiently small $\bar{\tau}$. Then, it follows from Proposition 1, Lemma 2 and Lemma 3 that for any $t\geq\bar{\tau}$,

$$\begin{aligned}&\frac{d}{dt}\left(\frac{1}{2}\sum_{i=1}^N|x_i(t)-x_c(t)|^2\right)\\ \geq\ &\sum_{i=1}^N(x_i^0-x_c)\cdot(v_i^0-v_c)-\|r\phi(r)\|_{L^1}\left(\frac{\|\alpha\|_\infty}{\int_0^{\tau_0}\alpha(s)ds}\|\tau'\|_{L^1}+1\right)-\frac{1-4k_0^2\lambda\|\phi\|_\infty\bar{\tau}}{4Nk}V^2(0)e^{-2kt}.\end{aligned}\tag{26}$$

Thus, if

$$\sum_{i=1}^N(x_i^0-x_c)\cdot(v_i^0-v_c)>\|r\phi(r)\|_{L^1}\left(\frac{\|\alpha\|_\infty}{\int_0^{\tau_0}\alpha(s)ds}\|\tau'\|_{L^1}+1\right),$$

from (26) there is a positive lower bound of $\frac{d}{dt}\sum_{i=1}^N|x_i-x_c|^2$ when $t$ is large enough. Combining this argument with (4), we know that $X(t)\to\infty$, so there is the non-existence of flocking. □

Specially, the condition (5) becomes $\sum_{i=1}^N(x_i^0-x_c)\cdot(v_i^0-v_c)>\|r\phi(r)\|_{L^1}$ as the delays approach zero. For model (1), (2), it is obvious to see that $X(t)$ is unbounded. It is coincident with Theorem 1 for C–S model without delay in [31].

## 4. A sufficient condition of flocking

For model (1), (2) with short range communication weights, a sufficient condition of the initial data was given to establish the non-flocking behavior in the above section. In this section, we investigate the flocking behavior for model (1), (2).

**Lemma 4.** *Let $0\leq\phi\in C_b(\mathbb{R}^+)$ be a decreasing function. Let $\tau\in[0,\bar{\tau}]$ satisfy that $e^{k\bar{\tau}}<2$. Then, the classical solution $\{(x_i,v_i)\}_{i=1}^N$ of model (1), (2) satisfies that for any $t\geq\bar{\tau}$,*

$$\frac{dV(t)}{dt}\leq-\frac{\lambda\phi(D(t))}{2}V(t)+32k_0^2\bar{\tau}^2\lambda^3\|\phi\|_\infty^3V(t),$$

*where $D(t)=\sup\{|x_k(s)-x_i(s)|:s\in[t-\bar{\tau},t],k\neq i\}$.*

**Proof.** It follows from (1) that for any $t\geq\bar{\tau}$,

$$\frac{d}{dt}\left(\sum_{i=1}^N|v_i-v_c|^2\right)=-\frac{\lambda}{Nh(t)}\sum_{i=1}^N\sum_{k=1}^N(v_k(t)-v_i(t))\int_{t-\tau(t)}^t\alpha(t-s)\phi_{ki}\cdot(v_k(s)-v_i(s))ds.\tag{27}$$

According to (27) and $v_k(s)-v_i(s)=v_k(t)-v_i(t)+v_k(s)-v_k(t)+v_i(t)-v_i(s)$, we have that

$$\begin{aligned}&\frac{d}{dt}\left(\sum_{i=1}^N\sum_{k=1}^N(v_k-v_i)^2\right)\\ =\ &-\frac{2\lambda}{h(t)}\sum_{i=1}^N\sum_{k=1}^N(v_k(t)-v_i(t))^2\int_{t-\tau(t)}^t\alpha(t-s)\phi_{ki}ds\\ &+\frac{4\lambda}{h(t)}\sum_{i=1}^N\sum_{k=1}^N\int_{t-\tau(t)}^t\alpha(t-s)\phi_{ki}(v_i(s)-v_i(t))(v_k(t)-v_i(t))ds,\qquad\forall t\geq\bar{\tau}.\end{aligned}$$

By the above equality and the Young inequality we get that for any $t \geq \bar{\tau}$

$$
\begin{aligned}
&\frac{d}{dt}\left(\sum_{i=1}^{N}\sum_{k=1}^{N}(v_k - v_i)^2\right) \\
\leq\ & -\frac{\lambda}{h(t)}\sum_{i=1}^{N}\sum_{k=1}^{N}(v_k(t) - v_i(t))^2 \int_{t-\tau(t)}^{t} \alpha(t-s)\phi_{ki}ds + \frac{4\lambda}{h(t)}\sum_{i=1}^{N}\sum_{k=1}^{N}\int_{t-\tau(t)}^{t}\alpha(t-s)\phi_{ki}(v_i(s) - v_i(t))^2 ds \\
\leq\ & -\frac{\lambda}{h(t)}V^2(t)\int_{t-\tau(t)}^{t}\alpha(t-s)\phi_{ki}ds + 4N\lambda\|\phi\|_\infty \sup_{s\in[t-\bar{\tau},t]}\sum_{i=1}^{N}(v_i(s) - v_i(t))^2, \qquad (28)
\end{aligned}
$$

since $h(t) = \int_0^{\tau(t)} \alpha(s)ds$. Then, we compute $\sum_{i=1}^{N}(v_i(s) - v_i(t))^2$. According to (1) and the Cauchy−Schwarz inequality, we have that for any $s \in [t - \bar{\tau}, t]$,

$$
\begin{aligned}
&(v_i(s) - v_i(t))^2 = \left(\int_s^t \dot{v}_i(u)du\right)^2 = \left|\int_s^t \frac{\lambda}{Nh(u)}\sum_{k=1}^{N}\int_{u-\tau(u)}^{u}\alpha(u-\omega)\phi_{ki}(v_k - v_i)(\omega)d\omega du\right|^2 \\
\leq\ & \frac{\lambda^2\|\phi\|_\infty^2}{N}\sum_{k=1}^{N}\left|\int_s^t\int_{u-\tau(u)}^{u}\frac{\alpha(u-\omega)(v_k - v_i)(\omega)}{h(u)}d\omega du\right|^2 \\
\leq\ & \frac{\lambda^2\|\phi\|_\infty^2}{N}\sum_{k=1}^{N}\left(\int_s^t\int_{u-\tau(u)}^{u}\frac{\alpha(u-\omega)|v_k - v_i|^2(\omega)}{h(u)}d\omega du\right)\left(\int_s^t\int_{u-\tau(u)}^{u}\frac{\alpha(u-\omega)}{h(u)}d\omega du\right).
\end{aligned}
$$

Following from (10), we get that

$$
\begin{aligned}
&\sum_{i=1}^{N}\sum_{k=1}^{N}\int_s^t\int_{u-\tau(u)}^{u}\frac{\alpha(u-\omega)|v_k - v_i|^2(\omega)}{h(u)}d\omega du \\
\leq\ & \sup_{w\in[t-2\bar{\tau},t]} V^2(w)\int_s^t\int_{u-\tau(u)}^{u}\frac{\alpha(u-\omega)}{h(u)}d\omega du \leq \left(k_0 e^{2k\bar{\tau}}V(t)\right)^2\int_s^t du \leq 16k_0^2\bar{\tau}V^2(t),
\end{aligned}
$$

since $h(t) = \int_0^{\tau(t)} \alpha(s)ds$. Combining the above two inequalities, we obtain that

$$
\begin{aligned}
&\sup_{s\in[t-\bar{\tau},t]}\sum_{i=1}^{N}(v_i(s) - v_i(t))^2 \\
\leq\ & \frac{\lambda^2\|\phi\|_\infty^2}{N}\cdot 16k_0^2\bar{\tau}V^2(t)\int_s^t\int_{u-\tau(u)}^{u}\frac{\alpha(u-\omega)}{h(u)}d\omega du \leq \frac{16k_0^2\bar{\tau}^2\lambda^2\|\phi\|_\infty^2}{N}V^2(t). \qquad (29)
\end{aligned}
$$

Thus, by (28) and (29) we have that

$$
\frac{d}{dt}\left(\sum_{i=1}^{N}\sum_{k=1}^{N}(v_k - v_i)^2\right) \leq -\frac{\lambda}{h(t)}V^2(t)\int_{t-\tau(t)}^{t}\alpha(t-s)\phi_{ki}ds + 64k_0^2\bar{\tau}^2\lambda^3\|\phi\|_\infty^3 V^2(t). \qquad (30)
$$

According to the definition of $D(t)$, from the decreasing of $\phi$ we can get that

$$
-\int_{t-\tau(t)}^{t}\alpha(t-s)\phi_{ki}ds \leq -h(t)\phi(D(t)).
$$

Thus, combining the above inequality with (30),

$$
2V(t)\frac{dV(t)}{dt} \leq -\lambda\phi(D(t))V^2(t) + 64k_0^2\bar{\tau}^2\lambda^3\|\phi\|_\infty^3 V^2(t),
$$

which completes the proof. □

**Theorem 2.** *Let $\tau \in [0, \bar{\tau}]$ satisfy that $e^{k\bar{\tau}} < 2$. Let $0 \le \phi \in C_b(\mathbb{R}^+)$ be a decreasing function. Assume that there exists $\gamma > 1$ satisfies that*

$$(\gamma - 1)(D(0) + 1)\phi(\gamma(D(0) + 1)) \ge \frac{8}{\lambda\sqrt{N}}V(0). \tag{31}$$

*Then the classical solution $\{(x_i, v_i)\}_{i=1}^N$ of model (1), (2) exhibits flocking behavior if $\bar{\tau}$ is small enough.*

**Proof.** According to the definition of $D(t)$ in Lemma 4, by the model of (1) we have that

$$|x_k(s) - x_i(s)| = |x_k(0) - x_i(0)| + \int_0^s |v_k(u) - v_i(u)|du.$$

Using the triangular inequality, we get that

$$\begin{aligned} & |v_k(u) - v_i(u)| \le |v_k(u) - v_c| + |v_i(u) - v_c| \\ \le \quad & 2\sqrt{\frac{|v_k(u) - v_c|^2 + |v_i(u) - v_c|^2}{2}} \le \sqrt{2}\sqrt{\sum_{i=1}^N |v_i(u) - v_c|^2} = \frac{V(u)}{\sqrt{N}}. \end{aligned} \tag{32}$$

From the above two inequalities and (13), we have that

$$\begin{aligned} D(t) \quad & \le \quad D(0) + \int_0^{\bar{\tau}} \frac{V(u)}{\sqrt{N}}du + \int_{\bar{\tau}}^t \frac{V(u)}{\sqrt{N}}du \le D(0) + 2\frac{V(0)}{\sqrt{N}}\bar{\tau} + \int_{\bar{\tau}}^t \frac{V(u)}{\sqrt{N}}du \\ & \le \quad D(0) + 1 + \int_{\bar{\tau}}^t \frac{V(u)}{\sqrt{N}}du, \end{aligned} \tag{33}$$

if $\bar{\tau}$ is small enough ($2\frac{V(0)}{\sqrt{N}}\bar{\tau} \le 1$). Note that $D(\bar{\tau}) \le D(0) + 1$. So for any $\gamma > 1$, there exists $t > \bar{\tau}$ such that $D(t) \le \gamma(D(0) + 1)$. Let $t_0$ be the largest time such that $D(t) \le \gamma(D(0) + 1)$.

Now, we prove $t_0 = \infty$. If not, $D(t_0) = \gamma(D(0) + 1)$. Then, from Lemma 4 we have that

$$\begin{aligned} \frac{dV(t)}{dt} \quad & \le \quad -\frac{\lambda\phi(\gamma(D(0) + 1))}{2}V(t) + 32k_0^2\bar{\tau}^2\lambda^3\|\phi\|_\infty^3 V(t) \\ & \le \quad -\frac{\lambda\phi(\gamma(D(0) + 1))}{4}V(t), \quad \forall\, t \in [\bar{\tau}, t_0], \end{aligned} \tag{34}$$

when $\bar{\tau}$ further satisfies that $128k_0^2\bar{\tau}^2\lambda\|\phi\|_\infty^3 \le \phi(\gamma(D(0) + 1))$. Following from (34), we get that for any $t \in [\bar{\tau}, t_0]$,

$$V(t) \le V(\bar{\tau}) \exp\left\{-\frac{\lambda\phi(\gamma(D(0) + 1))}{4}(t - \bar{\tau})\right\}. \tag{35}$$

Combining the above inequality with (33),

$$D(t) < D(0) + 1 + \frac{V(\bar{\tau})}{\sqrt{N}}\frac{4}{\lambda\phi(\gamma(D(0) + 1))} \le D(0) + 1 + \frac{V(0)}{\sqrt{N}}\frac{8}{\lambda\phi(\gamma(D(0) + 1))}, \quad \forall\, t \in [\bar{\tau}, t_0].$$

Thus, from assumption (31) we get that $D(t_0) < \gamma(D(0) + 1)$, which is in contradiction with $D(t_0) = \gamma(D(0) + 1)$. Thus, $t_0 = \infty$, and then (35) holds for any $t \ge \bar{\tau}$. □

**Remark 2.** *There does exist a sufficiently large $\gamma$ such that assumption (31) holds, if $r\phi(r) \to \infty$ as $t \to \infty$. A typical example is that $\phi(r) = (1 + r^2)^{-\beta/2}$ with $\beta \in [0, 1)$.*

## 5. Simulations

**Example 1.** Similarly as before, we first investigate the non-flocking behavior. Roughly, let $\alpha(t) = 1$, $N = 6$, $d = 3$, $\tau(t) = \frac{1+t}{2+t}$, $\phi(|x_k - x_i|) = \frac{10}{(1+|x_k-x_i|^2)^\beta}$, $\beta = 0.1$. But $\phi$ may not be a polynomial function in this paper. The initial positions and the initial velocities are

$$
\begin{aligned}
x_{10} &= (0.1, -0.5, 0.5), \quad v_{10} = (1, -1, 1),\\
x_{20} &= (0.2, 0, 0.3), \quad v_{20} = (-100, -5, 4),\\
x_{30} &= (0, 0.2, 0.1), \quad v_{30} = (0, 20, 40),\\
x_{40} &= (0.6, 0.5, 0.2), \quad v_{40} = (-2, 1, 3),\\
x_{50} &= (0.1, 0.2, 0.2), \quad v_{50} = (6, 200, 3),\\
x_{60} &= (0.8, 0.3, 0.1), \quad v_{60} = (80, 4, 2).
\end{aligned}
$$

It is easy to compute that the above initial condition satisfies (5) in Theorem 1. The velocities distance $v_i(t) - v_c(t)$ are shown in Fig.1(b). As expected, the system appears the non−flocking behavior. The simulation results successfully validate our theoretical analysis about Theorem 1.

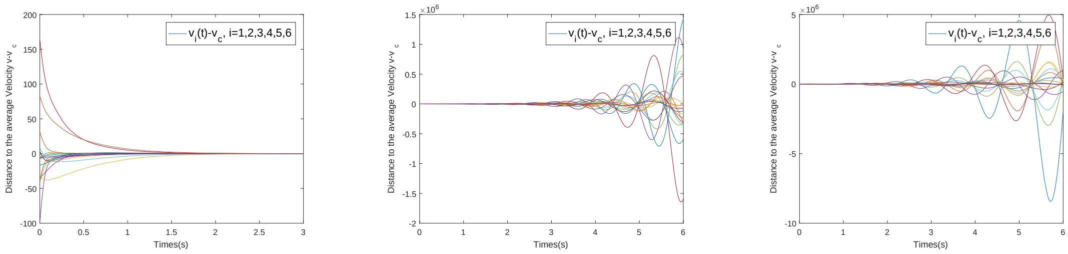
In addition, the velocity distances $v_i(t) - v_c(t)$ are shown in Fig.1(a)(b)(c)(for $\tau(t) = \frac{1+t}{10(2+t)}$, $\tau(t) = \frac{1+t}{2+t}$ or $\tau(t) = \frac{10(1+t)}{2+t}$ respectively) for the case of the critical exponent $\beta = 0.1$. From the above simulation results, we can know that $\tau(t)$ is a positive influence on the non−flocking behavior. The results are consistent with Theorem 1.

**Example 2.** For the behavior of flocking, we choose the following data: $\alpha(t) = 1$, $N = 3$, $d = 3$, $\tau(t) = \frac{1+t}{2+t}$, $\phi(|x_k - x_i|) = \frac{1}{(1+|x_k-x_i|^2)^\beta}$, $\beta = 0.1$, the initial positions and the initial velocities are

$$
\begin{aligned}
x_{10} &= (1, -5, 5), \quad v_{10} = (1, 1, 0),\\
x_{20} &= (-2, 3, 3), \quad v_{20} = (-2, -0.5, 0),\\
x_{30} &= (5, 2, 1), \quad v_{30} = (1, -2, 0).
\end{aligned}
$$

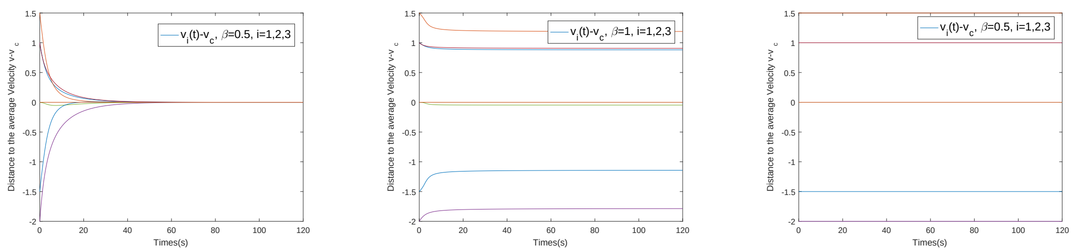
They satisfy the condition of Theorem 2. The velocities distance $v_i(t) - v_c(t)$ are shown in Fig.3(b). It is easy to see that there exists flocking, so the results successfully validate Theorem 2. Compared with the previous case (see Fig.1(b)), it seems that the flocking behavior is sensitive to the initial data. If initial data satisfy the condition of Theorem 2, there exists flocking. It exhibits non-flocking if initial data satisfy the condition (5).

Similar to Example 1, we change the value of $\beta$ in order to investigate the influence of $\phi$ on the flocking behavior. The simulation results are given in Fig.2(a)(b)(c)(for $\beta = 0.5$, $\beta = 1$ or $\beta = 10$ respectively). It is easy to see that $\phi$ promotes flocking, it is in keeping with the sufficient condition (31) in Theorem 2. It is also consistent with the classical results by Cucker and Smale in 2007 (see [1]). In addition, we can change the value of $\tau(t)$, then we obtain the following Fig.3(a)(b)(c)(for $\tau(t) = \frac{1+t}{10(2+t)}$, $\tau(t) = \frac{1+t}{2+t}$ or $\tau(t) = \frac{10(1+t)}{2+t}$ respectively). From the simulation results, we can discover that the flocking behavior is closely related to delay $\tau(t)$ and $\tau(t)$ is a negative influence on the flocking behavior. From this perspective, it verifies our conclusion from Theorem 2.



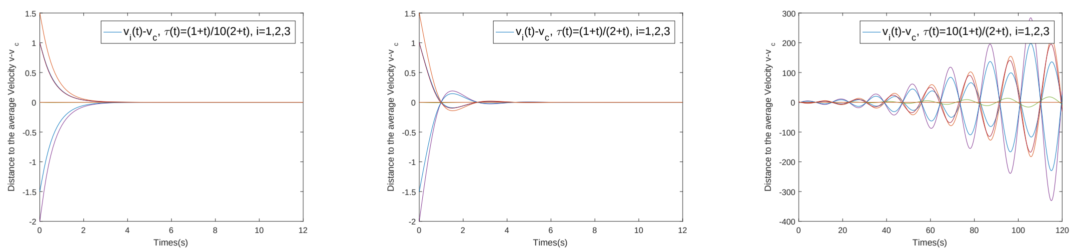

(a) $\tau(t) = \frac{1+t}{10(2+t)}$ (b) $\tau(t) = \frac{1+t}{2+t}$ (c) $\tau(t) = \frac{10(1+t)}{2+t}$

Figure 1: The velocity distance $v_i(t) - v_c(t)$ when $\tau(t) = \frac{1+t}{10(2+t)}$, $\tau(t) = \frac{1+t}{2+t}$ or $\tau(t) = \frac{10(1+t)}{2+t}$.




(a) $\beta = 0.5$ (b) $\beta = 1$ (c) $\beta = 10$

Figure 2: The velocity distance $v_i(t) - v_c(t)$ when $\beta = 0.5, \beta = 1$ or $\beta = 10$.




(a) $\tau(t) = \frac{1+t}{10(2+t)}$ (b) $\tau(t) = \frac{1+t}{2+t}$ (c) $\tau(t) = \frac{10(1+t)}{2+t}$

Figure 3: The velocity distance $v_i(t) - v_c(t)$ when $\tau(t) = \frac{1+t}{10(2+t)}$, $\tau(t) = \frac{1+t}{2+t}$ or $\tau(t) = \frac{10(1+t)}{2+t}$.

## 6. Conclusion

In this paper, we obtained sufficient conditions of non−flocking and flocking for the C−S model with distributed delays. It was the key to estimate an inequality of the position variance such that $X(t)$ is unbounded. A significant basis of our approach was that the non−existence of the asymptotic flocking is equivalent to the unboundedness of $X(t)$. Furthermore, by estimating an inequality of the derivative of $V(t)$, we obtained a sufficient condition that velocity decays to zero at the rate of exponential growth and the upper bound of position is bounded when $t$ is large enough. Above all, the sufficient conditions depend on initial data, communication weight and time delays according to simulation results and the main theorems.

## 7. Appendix

**Proof of Lemma 1.** By the Hölder inequality and (27), we have that for any $t \geq \bar{\tau}$,

$$
\begin{aligned}
&\frac{d}{dt}\left(\sum_{i=1}^{N}|v_i - v_c|^2\right)\\
\geq\ & -\frac{\lambda\|\phi\|_\infty}{Nh(t)}\int_{t-\tau(t)}^{t}\alpha(t-s)\left(\sum_{i=1}^{N}\sum_{k=1}^{N}|v_k(t)-v_i(t)|^2\right)^{\frac{1}{2}}\left(\sum_{i=1}^{N}\sum_{k=1}^{N}|v_k(s)-v_i(s)|^2\right)^{\frac{1}{2}}ds\\
\geq\ & -\frac{\lambda\|\phi\|_\infty}{Nh(t)}\int_{t-\tau(t)}^{t}\alpha(t-s)V(s)V(t)ds.
\end{aligned}
$$

From (4), we have that

$$\frac{d}{dt}V(t) \geq -\lambda\|\phi\|_\infty \sup_{s\in[t-\tau(t),t]} V(s), \tag{36}$$

since $h(t) = \int_0^{\tau(t)}\alpha(s)ds$. Then, by (36) and (12) we have that

$$V'(0) \geq -\lambda\|\phi\|_\infty\bar{\tau}k_0V(0).$$

By the continuous differentiability of $V(t)$ and the definition of $k$ (12) we obtain that $V'(t) > -kV(t)$ holds in some time intervals. Define that

$$t_0 = \sup\{t \geq 0 : V'(s) > -kV(s),\ \forall\ s \in [0,t)\}.$$

Now, we show that $t_0 = \infty$. If $t_0$ is finite, we have that

$$V'(t) > -kV(t),\ \ \forall\ t \in [0,t_0), \tag{37}$$

and

$$V'(t_0) = -kV(t_0). \tag{38}$$

Following from (37) we obtain that

$$V(s) < e^{k(t-s)}V(t),\quad \forall\ t_0 > t \geq s \geq 0. \tag{39}$$

When $s \in [-\bar{\tau}, 0)$, by the definition of $k_0$ and inequality (37) we obtain that for any $t_0 > t \geq 0 > s \geq -\bar{\tau}$

$$V(s) \leq k_0V(0) < k_0e^{kt}V(t) < k_0e^{k(t-s)}V(t).$$

Combining the above two inequalities, for any $t \in [0, t_0)$ we have that

$$V(s) < k_0 e^{k\bar{\tau}} V(t). \tag{40}$$

Using (36) again, we can obtain from the above inequality and the assumption of $\bar{\tau}$ that

$$V'(t) > -\lambda\bar{\tau}\|\phi\|_\infty k_0 e^{k\bar{\tau}} V(t), \quad \forall\, t \in [0, t_0).$$

At the same time, we can use (39) to deduce that

$$V(t_0) \geq k_0^{-1} e^{-kt_0} V(0) > 0.$$

Combining the above two estimates with $e^{k\bar{\tau}} \leq 2$ we get that

$$V'(t_0) \geq -\lambda\bar{\tau}\|\phi\|_\infty k_0 e^{k\bar{\tau}} V(t_0) = -kV(t_0).$$

Thus, $t_0 = \infty$, and $V'(t) > -kV(t)$ for any $t \geq 0$. Then, we can easily obtain (10) and (11) with $t \geq s$. For (11) with $t < s$, we can use a similar computation of (36) to get that

$$\frac{d}{dt}V(t) \leq \lambda\|\phi\|_\infty \sup_{s\in[t-\tau(t),t]} V(s).$$

By (11) with $t \geq s$ we get that

$$\frac{d}{dt}V(t) \leq k_0\lambda\|\phi\|_\infty \bar{\tau} e^{k\bar{\tau}} V(t) < kV(t),$$

which yields (11) for the case of $t < s$. □

**Proof of Lemma 2.** From the Hölder inequality we have that for any $t \geq \bar{\tau}$

$$\begin{aligned}
&\frac{d}{dt}\left(\sum_{i=1}^N (x_i - x_c)\cdot(v_i - v_c)\right)\\
=\ & \sum_{i=1}^N |v_i - v_c|^2 - \frac{\lambda}{2Nh(t)}\sum_{i=1}^N\sum_{k=1}^N \int_{t-\tau(t)}^t \alpha(t-s)\phi_{ki}(v_k(s) - v_i(s))(x_k(t) - x_i(t))ds\\
\geq\ & \frac{1}{2N}\sum_{i=1}^N\sum_{k=1}^N |v_k - v_i|^2 - \frac{\lambda\|\phi\|_\infty}{2Nh(t)}\int_{t-\tau(t)}^t \alpha(t-s)\left(\sum_{i=1}^N\sum_{k=1}^N |v_k - v_i|^2\right)^{\frac{1}{2}}\left(\sum_{i=1}^N\sum_{k=1}^N |x_k(t) - x_i(t)|^2\right)^{\frac{1}{2}} ds\\
=\ & \frac{1}{2N}\left(V^2(t) - \lambda\|\phi\|_\infty X(t)\int_{t-\tau(t)}^t V(s)ds\right).
\end{aligned}$$

Thus, we can obtain from (13) that

$$\begin{aligned}
&\sum_{i=1}^N (x_i(\bar{\tau}) - x_c)\cdot(v_i(\bar{\tau}) - v_c)\\
\geq\ & \sum_{i=1}^N (x_i^0 - x_c)\cdot(v_i^0 - v_c) + \frac{1}{2N}\int_0^{\bar{\tau}}\left(V^2(t) - \lambda\|\phi\|_\infty X(t)\int_{t-\tau(t)}^t V(s)ds\right)dt\\
\geq\ & \sum_{i=1}^N (x_i^0 - x_c)\cdot(v_i^0 - v_c) + \frac{\bar{\tau}}{2N}\left(\frac{1}{4}V^2(0) - \frac{\lambda\|\phi\|_\infty(X(0) + 2\bar{\tau}V(0))V(0)}{2k}\right)
\end{aligned}$$

since (11) gives that

$$\int_{t-\tau(t)}^{t} V(s)ds \leq V(t) \int_{t-\tau(t)}^{t} e^{k|t-s|}ds \leq \frac{V(t)}{k}.$$

□

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