Abscissa minimization for self-stability of bicycles and nonholonomic acceleration when riding out of the saddle

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Summary. The model of a two-mass-skate (TMS) bicycle is reconsidered from the point of view of optimization of its self-stability, i.e. asymptotic stability of its vertical position in a uniform straight motion with respect to small perturbations of its leaning and steering angles. The critical Froude number for the onset of the self-stability is found explicitly. Minimization of the spectral abscissa is then performed yielding a new scaling low that links together the geometric and the mass parameters of the TMS bicycle. The scaling law fits well the relations between the geometric and mass parameters of a recent experimental realization of the TMS bicycle. Modification of the TMS model by including into it a mass moving periodically in the direction perpendicular to the frame of the bicycle is discussed as a step to understand nonholonomic acceleration of bicycles when riding out of the saddle.

Self-stability of bicycles and its optimization

The bicycle is easy to ride, but surprisingly difficult to model. Refinement of the mathematical model of a bicycle has continued over the last 150 years with contributions from Rankine, Boussinesq, Whipple, Klein, Sommerfeld, Appel, Synge and many others [1, 2, 3]. The canonical, nowadays commonly-accepted model goes back to the 1899 work by Whipple [1, 4, 5]. The Whipple bike is a system consisting of four rigid bodies with knife-edge wheels making it non-holonomic, i.e., requiring for its description more configuration coordinates than the number of its admissible velocities.

Self-stability of a riderless bicycle is a well-known empirical phenomenon that can be easily reproduced in an experiment with the majority of known practical designs of bicycles. Namely, a bicycle in its uniform motion along a straight path keeps the vertical position of the plane of its frame under small perturbations if its forward speed is high enough. Moreover, perturbations of its leaning and steering angles die out, so that one can say about asymptotic self-stability of the vertical position of the bicycle. The fact of the asymptotic self-stability has been theoretically supported already by Whipple [1]. The self-stability has a number of important practical implications. For instance, the bicycle designs that do not present the self-stability are difficult for a person to ride; in other words, more stable bikes handle better. Hence, deeper understanding of the passive stabilization can provide new principles for the design of more safe and rideable bicycles, including compact and foldable models. Furthermore, it is expected to play a crucial part in formulating principles of the design of energy-efficient wheeled and bipedal robots [6].

However, the theoretical explanation of the self-stability has been highly debated throughout the history of bicycle dynamics to such an extent that a recent news feature article in Nature described this as “the bicycle problem that nearly broke mathematics” [7]. The reason as to why “simple questions about self-stabilization of bicycles do not have straightforward answers” [8] lies in the symbolical complexity of the Whipple model that contains 7 degrees of freedom and depends on 25 physical and design parameters [2]. In recent numerical simulations [1, 4, 5], self-stabilization has been observed for some benchmark designs of the Whipple bike. These results suggested further simplification of the model yielding a reduced model of a bicycle with vanishing radii of the wheels (which are replaced by skate blades), known as the two-mass-skate (TMS) bicycle [3, 9, 10]. Despite the self-stable TMS bike having been successfully realized in recent laboratory experiments [3], its self-stability still awaits a theoretical explanation. In this lecture we find explicitly the critical Froude number for the onset of-self-stability of the TMS bicycle and will show how minimization of the spectral abscissa [11, 12] allows one to find hidden symmetries in the model, suggesting further reduction of the parameter space and, finally, providing explicit relations between the parameters of stability-optimized TMS bikes [13, 14].

Nonholonomic acceleration due to honking or riding out of the saddle

Honking means cycling out of the saddle. Cyclists use this way of riding to accelerate. In this process, the body of the cyclist moves rhythmically side to side with respect to the plane of the bicycle, while the frame of the bicycle rocks or sways with respect to the vertical position. It is known, however, that the commonly accepted modern bicycle models, stemming from the Whipple model of 1899, are non-holonomic and conservative. On one hand, this implies the conservation of energy, and on the other, non-conservation of the phase volume, which results in the possibility of the asymptotic stability of a straight vertical position of a bicycle that is riding along a straight path if the forward velocity is high enough, which is a well-known empirical fact [2, 3]. A natural question is: where the energy of dying leaning and steering motions flows? Actually, the energy in the lean and steer oscillations is transferred via a nonlinear coupling to the forward speed rather than being dissipated, as it was shown in numerical simulations in [2]. This effect suggests a method of acceleration of the bicycle due to periodic movements of a mass in the direction perpendicular to the plane of the frame of the bicycle. It can therefore be conjectured that when cycling out of the saddle, the periodical movement of the cyclist’s mass pumps energy into the forward motion leading to the desired acceleration.

Recently, acceleration due to a periodic motion of an internal mass has been theoretically discovered in a non-holonomic

Chaplygin sleigh model [15, 16] and linked to the mechanism of acceleration of particles moving between periodically oscillating walls in the Fermi-Ulam model [17]. In the present talk we will discuss the ways of extending the model of a TMS bicycle to take into account a periodically moving mass for investigation of the nonholonomic Fermi-like acceleration during cycling out of the saddle.

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